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Completing Partial Latin Squares

Abstract. *The paper gives a survey of completion results for symmetric and unsymmetric partial latin squares. Several embedding results are mentioned, but the emphasis is on proper completion where, given a partial latin square of side n, one looks for a completion to a latin square of the same side.*

For latin squares with no symmetry required we prove a strengthening of the known results concerning the Evans conjecture, which was proved to be true in 1981.

We then state some new results about the corresponding problem for symmetric latin squares and describe their proofs briefly; complete proofs will be given elsewhere.

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1. Introduction

The subject of this paper belongs to the branch of mathematics called *combinatorics*. This area is concerned with arranging, counting, and choosing from a number of objects. Very often only a finite number of objects are considered at a time, and the discrete nature of the subject gives it a striking feature not so common in other parts of mathematics: frequently a deep mathematical problem can be explained in very simple terms, easy to understand for anybody who cares to listen.

Probably the most famous problem sharing this particular beauty of combinatorics is the so-called *four colour problem:* Is it true that any map in the plane or on a sphere can be coloured so that any pair of countries with a common boundary always get different colours, using altogether at most four colours? (It must be required that each country consists of just one connected region). The four colour problem was unsolved for more than a century, though many skilled mathematicians have worked very hard on it (and so have many amateur mathematicians!) It has now been established that four colours do suffice to colour any map; a proof was announced in 1976 and published in 1977 (Appel & Haken 1976 and 1977, Appel, Haken & Koch 1977). But the proof depends heavily on the use of a computer, and so the 'simple' four colour problem still puzzles the minds of combinatorialists hoping to find a more direct proof.

This article is about a concept in combinatorics called *latin squares* (completely unrelated to the four colour problem), a topic which in the course of time also has contained some simply stated problems that turned out to be extremely difficult. Consider the following example.

Let n be a positive integer. Given an n×n array (chessboard) in which n-1 of the cells (squares) are filled with one of the integers 1,…,n, so that no integer occurs twice in any row or column, is it always possible to fill all the remaining cells to obtain an n×n array in which *each* of the numbers 1,…,n occurs *exactly once* in each row and column?

This question was first posed by T. Evans (Evans 1960), and the assumption of an affirmative answer became known as *the Evans Conjecture.* Despite the fact that the problem received much attention, and many partial solutions were published, no complete solution was given until 1981, when B. Smetaniuk proved that the Evans Conjecture is true, and so the answer to the above question is indeed yes (Smetaniuk 1981).

The present paper is closely related to the Evans Conjecture, but before continuing the discussion we have better put the topic in its proper context of latin squares.

A *partial latin square* of side n on the symbols $s_1,\ldots,s_n$ is an n×n matrix of cells in which each cell either is empty or contains one of the symbols $s_1,\ldots,s_n$, and, furthermore, no symbol occurs twice in any row or twice in any column. It is a *latin square* if there are no empty cells. Thus in a latin square each symbol must occur exactly once in each row and column. We shall almost always assume that the set of symbols is $\{1,\ldots,n\}$. Figure 1 shows two latin squares and two further partial latin squares, all of side 6.

With the above definitions, the Evans Conjecture can be formulated like this: Any partial latin square of side n with at most n-1 non-empty cells

Figure 1

1	3	4	6	2	5
3	6	5	1	4	2
4	5	1	2	3	6
6	1	2	3	5	4
2	4	3	5	6	1
5	2	6	4	1	3

a

1	2	4	5	6	3
5	4	6	3	2	1
2	1	5	6	3	4
6	3	2	4	1	5
4	6	3	1	5	2
3	5	1	2	4	6

b

1	2	4			
5	4	6			
2	1	5			
			4		
				5	
					6

c

1	2	3	4		
				5	
				6	

d

can be completed to a latin square of side n. The same conjecture has been stated on at least two other occasions (Klarner 1970; Dénes 1974).

After B. Smetaniuk's proof, an independent proof was published by A. J. W. Hilton and the present author proving, however, a stronger result (Andersen & Hilton 1983). T. Evans arrived at his conjecture partly because he could find no counterexamples, and partly because it is easy to find examples of a partial latin square of side n with n non-empty cells which *cannot* be completed to a latin square of side n (Figure 1d is such an example). The paper by Andersen and Hilton actually contained a *complete characterization* of those partial latin squares of side n with n non-empty cells that cannot be completed to a latin square of side n.

In Section 5 of this paper we extend the characterization to include all those partial latin squares of side n with $n+1$ nonempty cells which cannot be completed to a latin square of side n.

It may be asked where the importance of determining such a further class of non-completable partial latin squares lies. The author believes that throughout mathematics determining extreme cases is very valuable and so, in particular, there is a large difference between knowing that squares with n non-empty cells can be completed *except* in a well-defined, non-empty class of situations and just knowing that squares with n-1 non-empty cells can be completed. Furthermore, by characterizing exceptions 'one step beyond' the extreme cases, a good deal more insight in the structure of the problem is gained (accordingly, we believe that the exceptions presented in Section 5 give a good idea about how non-completable squares can be obtained if yet more non-empty cells are introduced). In the present case it is also useful for the proof of one of the main results of Section 6.

Another possible question could be whether a 'nice' characterization of non-completable squares with *any* number of non-empty cells is obtainable. The answer is almost certainly no, as the problem of completing partial latin squares is known to be *NP-complete* and so belongs to a class of problems that are not expected to have 'easy solutions'. We discuss this in more detail in the next section.

While on the subject of questions, a simple, but natural one is: What is the purpose of latin squares in general and of completing partial latin squares in particular? It is beyond the scope of this paper to discuss applications of latin squares at any great length, but we can single out the subject of *design of experiments* as probably the main field concerned with practical applications of latin squares. They are used in some situations where one wants to gather data for statistical analysis, and the purpose of the latin squares is to eliminate the effect of certain systematic differences from the data. We mention briefly two examples of slightly differing nature:

1. Sheer planning purposes:
   Suppose that we want to test the effect of n different diets on the milk yield of cattle. We select n cows and n time periods and use a latin square of side n with a row for each cow and a column for each time period. Then, if, say, the cell common to the row of cow 1 and the column of period 2 contains the symbol 3 it means that cow 1 is given diet 3 in period 2, etc. With proper handling of the data, this would eliminate differences between cows and time periods, as each diet is tested on each cow and in each time period.
2. As a 'physical' latin square:
   If n varieties of a crop are to be tested on a rectangular field, it will often be advantageous to superimpose an $n \times n$ latin square on the field, thus dividing it into n^2 smaller rectangular plots all having the same size, and growing crop 1 in the n plots corresponding to the occurrences of 1 in the latin square, etc. There is a good chance that this will eliminate yield differences due to fertility differences of the plots.

In both of the above examples, further properties of the latin squares (such as *row completeness*) would be desirable, so the description here is much simplified. We must omit further discussion and refer the reader to some of the many books on the subject (Dénes & Keedwell 1974; Fisher 1935; Cochran & Cox 1950; Cox 1958; The Open University 1981). It is not hard to imagine situations like the examples above, where it would be convenient to have certain entries of the latin squares fixed *a priori;* this is where completion of partial latin squares comes in.

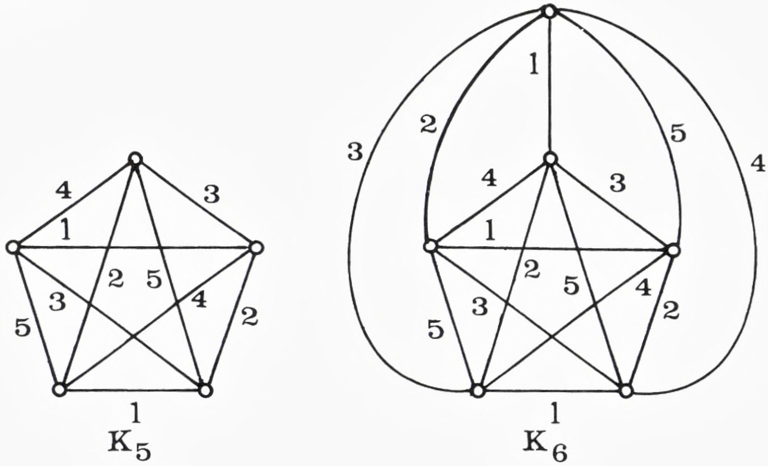
An even simpler example of this is obtained when considering the completion of a partial latin square as a *time-tabling* problem in the following way. We have two sets S_1 and S_2, each consisting of n persons, and we must schedule one meeting between each person from S_1 and each person from S_2, all meetings taking place within n time periods. The set S_2 could consist of participants of a course and S_1 be the teachers of the course with the meetings corresponding to examinations. Such a schedule is readily provided by a latin square of side n with the rows corresponding to S_1, the columns to S_2 and the entries 1,…,n to the time periods. If some of the meetings have to take place at prescribed times, we have our completion problem. It may be argued that in a situation such as the above, it is unlikely that the number of teachers is equal to the number of participants; it follows, however, from one of the simplest completion results of Section 3 that our results will cover the case of $|S_1| \neq |S_2|$ also (for any set A, $|A|$ denotes the number of elements of A).

If we have just one set S of persons and we want to schedule one meeting between each pair of persons we can use a *symmetric latin square*. We have defined latin squares as matrices, and so it is customary to enumerate the rows from the top and the columns from the left, so that cell (i,j) is the cell common to the ith row from the top and the jth column from the left. We say that a partial latin square of side n is *symmetric* if, whenever one of the cells (i,j) and (j,i) is non-empty then so is the other, and they have the same entry ($1 \leqslant i < j \leqslant n$). Figure 1a shows a symmetric latin square. The *diagonal* is the set of cells $\{(i,i) | 1 \leqslant i \leqslant n\}$. In a symmetric latin square of odd side each symbol occurs exactly once on the diagonal, and in a symmetric latin square of even side each symbol occurs an even number of times on the diagonal.

We can use a symmetric latin square for scheduling meetings between each pair from a group of n persons (for example as required for a round-robin tournament) by letting persons i and j meet in time period k, where k is the entry of cells (i,j) and (j,i). This will require n time periods although each person has just n-1 meetings; in each time period occurring on the diagonal one or more persons have no meetings, person i having time period k off where k is the entry of cell (i,i). When scheduling pairwise meetings for an odd number of persons, clearly one person *has* to be unpaired in each period, and so n time periods are indeed needed altogether. If n is even, however, the symmetric latin square again brings n time periods into the schedule, where only n-1 are needed, as we shall see. One way of getting round this is to require that all cells on the diagonal of the symmetric latin square contain the entry n; then no meeting will be scheduled for period n, so only n-1 periods are used. But perhaps a more natural way of looking at round-robin tournaments with an even number of participants is from the point of view of *complete graphs*.

A graph G = (V,E) consists of a set V called *vertices* and a set E of *edges*, which are pairs of distinct vertices. An edge $e = \{V_1, V_2\}$ is said to *join* vertices V_1 and V_2, and to be *incident* with V_1 and V_2, and we write $e = V_1V_2$. We also say that V_1 and V_2 are *neighbours*. A *complete* graph is a

Figure 2



graph in which each pair of vertices are joined by an edge, and the complete graph with n vertices is denoted by K_n. Figure 2 shows drawings of K_5 and K_6. The numbers on the edges are explained below.

An *edge-colouring* of a graph G with k colours is an assignment of one of k colours to each edge of G such that edges incident with the same vertex always have different colours. We shall always use the set of 'colours' {1,...,k}. Each of the graphs K_5 and K_6 of Figure 2 has been given an edge-colouring with 5 colours. The *chromatic index* q(G) of G is the least k for which G has an edge-colouring with k colours. A famous theorem states that if d(G) is the *maximum degree* of G (the largest number of edges incident with any one vertex), then q(G) is either d(G) or d(G)+1 (Vizing 1964). It is wellknown that

$$q(K_n) = \begin{cases} n-1 & \text{if n is even,} \\ n & \text{if n is odd.} \end{cases}$$

Thus the edge-colourings given to K_5 and K_6 in Figure 2 are minimal, i.e. with as few colours as possible.

If the vertices of a graph G correspond to persons, then clearly an edge-colouring of G provides a schedule for meetings between all pairs joined by an edge: each meeting between a pair takes place in the time period designated by the colour of the edge joining the pair. If only q(G) colours are used for the edge-colouring, then as few time periods as possible are used.

The round-robin tournament problem then corresponds to constructing an edge-colouring of K_n with $q(K_n)$ colours, possibly with the colour of some edges prescribed. We consider this problem, as well as that of completing partial symmetric latin squares, in Section 6, where we present two new results and some corollaries that we have recently obtained with A. J. W. Hilton and E. Mendelsohn, respectively. We only sketch the proofs; they are too complicated to be included here and will appear elsewhere.

Given an edge-colouring of K_n with $q(K_n)$ colours it is easy to construct a symmetric latin square of side n. Let the vertices be $V_1,\ldots,V_n$ and place the colour of the edge V_iV_j in cells (i,j) and (j,i), $1\leqslant i<j\leqslant n$. If n is odd, place the colour not occurring at the vertex V_i in cell (i,i) and if n is even, place the colour n in all cells (i,i), $1\leqslant i\leqslant n$. This is easily seen to be a symmetric latin square of side n. Actually, if n is even we can obtain a symmetric latin square of side n-1 also by deleting the last row and column and placing the entry of cell (n,i) in cell (i,i) instead, $1\leqslant i\leqslant n-1$.

We close this introductory section by noting that also the problem of completing partial latin squares with no symmetry required has a graph analogue. A graph $G = (V,E)$ is called *bipartite* if V can be partitioned into two disjoint sets L and R such that each edge joins a vertex of L to a vertex of R. If *each* vertex from L is joined to *each* vertex from R then G is a *complete bipartite* graph, and it is denoted by $K_{m,n}$ where $m = |L|$ and $n = |R|$. For any bipartite graph G, $q(G) = d(G)$ (König 1936). A latin square of side n is equivalent to an edge-colouring of $K_{n,n}$ with n colours simply by considering cell (i,j) to be an edge joining vertices ℓ_i and r_j where $L = \{\ell_1,\ldots,\ell_n\}$ and $R = \{r_1,\ldots,r_n\}$, and $1\leqslant i\leqslant n, 1\leqslant j\leqslant n$. So completing a partial latin square of side n to a latin square of side n is equivalent to finding an edge-colouring of $K_{n,n}$ with n colours, with the colour of some edges being prescribed.

The author is aware that this introduction has been much more diffuse (and longer!) than is usual for a mathematical research paper, even for a survey paper. This is due to the context in which this paper appears, a context enhancing the possibility that non-mathematicians may stumble upon the paper. It has been the purpose of the introduction to give such non-professionals an opportunity of getting an impression of the subject and of getting some idea of the contents of the paper. We shall *not* keep up this style in the remainder of the paper. We intend to survey part of the enormous amount of material on completions of partial latin squares, with emphasis on the results already mentioned in this introduction, and from now on we shall use standard terminology without defining all the concepts used. Quite often the definitions can be found in the references that we give.

2. The complexity of completing partial Latin Squares

For the theory of NP-completeness we refer the reader to one of the excellent books on the subject (Garey & Johnson 1979). Here we remark that the class of decision problems which can be solved by polynomial time algorithms is called P, and that $P \subseteq NP$, where NP is a class of problems containing several apparently very difficult problems unlikely to be solvable in polynomial time (here, a polynomial time algorithm is considered 'good', problems which cannot be solved in polynomial time are considered 'hard', and a 'problem' should not be confused with an 'instance' of a problem). It is not known whether $P = NP$, but it is considered extremely improbable.

A subclass of NP consists of the *NP-complete* problems. Any NP-complete problem has the property that if it can be solved in polynomial time then $P = NP$. So no NP-complete problem is expected to be solvable by a polynomial time algorithm.

C. J. Colbourn has been interested in the complexity of completing partial latin squares. He first proved that completing partial *symmetric* latin squares is NP-complete (and used this to prove that embedding partial Steiner triple systems is NP-complete as well) (Colbourn 1983). Shortly afterwards, Colbourn proved that also completing partial latin squares is NP-complete (Colbourn 1984). We sketch his proof for partial latin squares and use that result to present a new, simpler proof for the symmetric case.

The investigation of the complexity of completing partial latin squares makes use of another link between latin squares and graph theory, different from the connection described in Section 1.

A graph $G = (V,E)$ is called *tripartite* if V can be partitioned into three mutually disjoint sets V_1, V_2 and V_3 such that the end-vertices of each edge of E are in distinct sets. If each pair of vertices from distinct sets are joined by an edge, G is said to be a *complete tripartite* graph, and it is denoted by $K_{\ell,m,n}$ where $\ell = |V_1|$, $m = |V_2|$ and $n = |V_3|$. It is easy to see that a latin square of side n is equivalent to a decomposition of $K_{n,n,n}$ into mutually edge-disjoint K_3's; just let V_1, V_2 and V_3 be the set of rows, columns and symbols respectively, and identify an occurrence of the symbol k in cell (i,j) with the K_3 with vertices corresponding to row i, column j and symbol k. This idea is exploited in the following.

It is known that the problem of determining whether a graph can be decomposed into mutually edge-disjoint K_3's is NP-complete (Holyer 1981a). Modifying the proof of this, Colbourn obtained:

Theorem 2.1. Deciding whether a tripartite graph can be decomposed into mutually edge-disjoint K_3's is NP-complete.

A tripartite graph is *uniform* if each vertex has the same number of neighbours in each of the vertex classes not containing it. If a tripartite graph is not uniform, then it is obvious that it cannot be decomposed into mutually edge-disjoint K_3's. So we have

Corollary 2.2. Deciding whether a uniform tripartite graph can be decomposed into mutually edge-disjoint K_3's is NP-complete.

Now let G = (V,E) be a tripartite graph with vertex classes $\{r_1,\ldots,r_x\}$, $\{c_1,\ldots,c_y\}$ and $\{s_1,\ldots,s_z\}$. A *latin framework* LF(G;s) for G, $s \geqslant \max\{x,y,z\}$, is a partial latin square of side s on symbols 1,...,s with the following properties:

(i) Cell (i,j) of LF(G;s) is empty if and only if $r_ic_j \in E$.
(ii) If $r_is_k \in E$ then k does not occur in row i of LF(G;s).
(iii) If $c_js_k \in E$ then k does not occur in column j of LF(G;s).

Colbourn proved that G always has a latin framework with $s = 2|V|$:

Theorem 2.3. Given a uniform tripartite graph G with n vertices, a latin framework LF(G;2n) can be produced in polynomial time.

With this, it is an easy matter to prove the main result of this section.

Theorem 2.4. Deciding whether a partial latin square of side n can be completed to a latin square of side n is NP-complete.

Proof. The problem is clearly in NP. To prove that it is NP-complete, by Corollary 2.2 it suffices to reduce the problem of decomposing a uniform tripartite graph into mutually edge-disjoint K_3's to the problem of completing a partial latin square, the reduction taking place in polynomial time. But this can be done by Theorem 2.3 and the observation that

Figure 3

$$S(P): \quad \begin{array}{|c|c|}\hline S_n & P \\ \hline P^T & S_n \\ \hline \end{array}$$

G can be decomposed into mutually edge-disjoint K_3's if and only if LF(G;2n) can be completed to a latin square of side 2n. We leave this argument to the reader (note that (i) and the uniformity of G imply that the converse of (ii) also holds: if k does not occur in row i, then $r_i s_k \in E$).

From Theorem 2.4 we get a new proof of the next theorem.

Theorem 2.5. Deciding whether a partial symmetric latin square of side n can be completed to a symmetric latin square of side n is NP-complete.

Proof. Obviously the problem belongs to NP. For any n, let S_n be any symmetric latin square of side n on symbols n+1,...,2n. If P is any partial latin square of side n on symbols 1,...,n, let S(P) be the partial symmetric latin square of side 2n on symbols 1,...,2n indicated in Figure 3.

Clearly S(P) can be constructed in polynomial time, and clearly S(P) can be completed to a symmetric latin square of side 2n if and only if P can be completed to a latin square of side n. Thus we have obtained a polynomial time reduction from completion with no symmetry to completion with symmetry, and Theorem 2.5 follows from Theorem 2.4.

It follows from the proof of Theorem 2.5 that completing partial symmetric latin squares is NP-complete even when restricted to the class of squares which are as in Figure 3, a class which at first sight might look rather limited.

Colbourn's original proof of Theorem 2.5 made use of the following result, which we include for completeness (Holyer 1981b):

Theorem 2.6. It is NP-complete to determine the chromatic index of a graph G. In fact, it is NP-complete even to determine the chromatic index of a graph which is regular of degree 3.

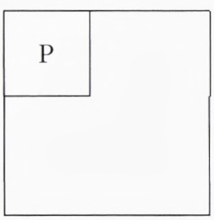
The moral of this section is that it is probably a desperate undertaking to try giving a simple characterization of those partial latin squares of side n that can be completed to a latin square of side n. So it seems reasonable that most of the research in the area has concentrated on determining families of partial latin squares that *can* be completed. The next sections contain many examples of such families. Let us make it clear, however, that when we prove, or state without proof, that a partial latin square of a certain type can be completed, we do not present an algorithm for actually doing so. As a general rule, our existence proofs are not very constructive.

3. Embedding results

We have seen that not all partial latin squares can be completed, in the sense that not all partial latin squares of side n can be completed to a latin square of side n. A natural question to ask then is whether a partial latin square of side n can always be completed to a latin square of side t for some $t > n$? In the same paper as where the Evans Conjecture was posed, Evans proved that the answer is affirmative, and that any $t \geqslant 2n$ will do (and for each $n \geqslant 4$ he gave examples where no $t < 2n-1$ works) (Evans 1960).

When we have a completion as above, we say that the partial latin square P can be *embedded* in the larger latin square S. Usually we think of P as being situated in the top left hand corner of S, as in Figure 4.

There are two ways of looking at the situation of Figure 4. One is as described above, a straightforward embedding of a partial latin square P of side n in a latin square S of side t. The other sees it as a completion of a partial latin square of side t, where all non-empty cells just happen to



S

example:

1	2	
		3
2	1	

P

1	2	4	6	5	3
5	4	3	2	6	1
2	1	5	3	4	6
3	6	2	5	1	4
4	3	6	1	2	5
6	5	1	4	3	2

S

Figure 4

occur inside an $n \times n$ subarray. The former seems the more appropriate, but allows only n distinct symbols to occur in P, whereas the latter would allow all t symbols to do so. We compromise and choose embedding terminology, but allow t symbols to occur in P. Hence we shall speak of a partial latin square *of side n on t symbols*.

As mentioned in the introduction, our main interest will be some completion theorems in the vein of the Evans Conjecture, and our survey of embedding results will be brief. The reader will be able to find more on this topic in the recent survey paper by C. C. Lindner (Lindner 1984).

Before we begin listing results, we need one more definition. Following Lindner, an *$r \times s$ latin rectangle on symbols $1,\ldots,n$* is an $r \times s$ array in which *each* cell contains an element of $\{1,\ldots,n\}$, such that each symbol occurs at most once in each row and column. Note that this also applies to the case $r=s$, so that an $r \times r$ latin rectangle has no empty cells, and yet it need not be a latin square!

If R is a latin rectangle on symbols $1,\ldots,n$ we let $R(i)$ denote the number of occurrences of the symbol i in R, $1 \leqslant i \leqslant n$. We first prove a lemma which contains a necessary condition for embedding that appears in a variety of situations.

Lemma 3.1. Let R be an $r \times s$ latin rectangle which is embedded in a latin square S of side n on symbols $1,\ldots,n$, and let D be as in Figure 5. Then $R(i) = r+s-n+D(i)$ for all i, $1 \leqslant i \leqslant n$.

Proof. Follows from the facts that for all i, $R(i)+B(i)=s$ and $B(i)+D(i) = n-r$.

If an $n \times n$ array S is partitioned as in Figure 5, and if R is $r \times s$, then the *diagonal outside R* is the set of cells $(r+1,s+1)$, $(r+2,s+2),\ldots,(r+n-s,n)$ if $r \leqslant s$, and the set of cells $(r+1,s+1),(r+2,s+2),\ldots,(n,s+n-r)$ if $r \geqslant s$. They all lie in D.

The strongest embedding result we have when no symmetry is required is the following (Andersen & Hilton 1983). It is concerned with embedding of a latin rectangle with the additional requirement that the diagonal outside R is prescribed (each symbol i must occur f(i) times on it). The Figures 1c and 1b give an example of such an embedding. However, if $r = s$ then for Theorem 3.2 to work at least one cell on the diagonal must be left unprescribed (Andersen, Häggkvist, Hilton & Poucher 1980).

Figure 5

S:

R	A
B	D

Theorem 3.2. Let R be an r×s latin rectangle on symbols 1,...,n, and for each i, $1 \leqslant i \leqslant n$, let f(i) be a non-negative integer such that $\sum_{i=1}^{n} f(i) \leqslant \min\{n-r, n-s\}$ with strict inequality if r=s. Then R can be embedded in a latin square of side n with symbol i occurring at least f(i) times on the diagonal outside R for all i, $1 \leqslant i \leqslant n$, if and only if

$$R(i) \geqslant r+s-n+f(i) \text{ for all } i,\ 1 \leqslant i \leqslant n.$$

From Theorem 3.2 it is easy to deduce some wellknown results.

Corollary 3.3. (Ryser 1951). An r×s latin rectangle R on symbols 1,...,n can be embedded in a latin square of side n if and only if

$$R(i) \geqslant r+s-n \text{ for all } i,\ 1 \leqslant i \leqslant n.$$

Proof. Put all f(i) equal to zero in Theorem 3.2.

Corollary 3.4. (Evans 1960). A partial latin square of side n on t symbols can be embedded in a latin square of side t for all $t \geqslant 2n$.

Proof. All empty cells of the partial latin square can be filled with one of the t symbols, as at most 2(n−1) symbols can be forbidden for a given cell. Now apply Corollary 3.3, where the condition is satisfied because $R(i) \geqslant 0 \geqslant 2n-t$.

The condition $t \geqslant 2n$ is best possible, as we remarked earlier.

Corollary 3.5. (Hall 1945). An r×n latin rectangle on n symbols can always be completed to a latin square of side n.

A time-table for all meetings between pairs of persons, one belonging to a set of r persons and the other to a set of n>r persons disjoint from the first set, in as few time periods as possible, corresponds to an r×n latin rectangle on n symbols (and to an edge-colouring of $K_{r,n}$ with n colours). Corollary 3.5 shows that trying to complete such a time-table with some preassigned entries is equivalent to trying to complete to a latin square of side n.

Theorem 3.2 does not hold with r=s and $\sum_{i=1}^{n} f(i) = n-r$, i.e. with completely prescribed diagonal. This is unfortunate, because a frequent requirement on a latin square is that it is *idempotent,* meaning that cell (i,i) contains the symbol i for all i. No necessary and sufficient condition for a latin rectangle to be embeddable in an idempotent latin square is known, which is valid in all cases. Problems concerning this have been extensively studied (Lindner 1971; Hilton 1973; Andersen 1982; Hilton & Rodger 1982; Rodger 1983; Andersen, Hilton & Rodger 1983; Bryant 1984). We state only two results, the first very important theorem due to C. A. Rodger giving a necessary and sufficient condition for the case r=s and $n \geq 2r+1$ (and $r \geq 10$).

Theorem 3.6. (Rodger 1984). Let R be an $r \times r$ latin rectangle on symbols 1,...,n where $r \geq 10$ and $n \geq 2r+1$. For each i, $1 \leq i \leq n$, let f(i) be a non-negative integer such that $\sum_{i=1}^{n} f(i) = n-r$. Then R can be embedded in a latin square of side n with each symbol i occurring f(i) times on the diagonal outside R for all i, $1 \leq i \leq n$, if and only if (i)-(iii) are satisfied.

(i) $R(i) \geq 2r-n+f(i)$ for all i, $1 \leq i \leq n$.
(ii) For all i, $1 \leq i \leq n$: if $R(i)=r$ then $f(i) \neq n-r-1$.
(iii) If R is a latin square and $n=2r+1$ then $\sum_{R(i)>0} f(i) \neq 1$.

For $n \geq 10$, the next theorem is a corollary of Theorem 3.6.

Theorem 3.7. (Andersen, Hilton & Rodger 1982). A partial idempotent latin square of side n can be embedded in an idempotent latin square of side t for all $t \geq 2n+1$.

When Theorem 3.7 first appeared it settled a long standing conjecture in the affirmative. The inequality $t \geq 2n+1$ is the best possible.

Turning now to embedding theorems for partial symmetric latin squares we first present a new result due to A. J. W. Hilton and the author. It is used in the proof of one of the main results of Section 6; in this paper we state both theorems without proofs. Because of the close connection between symmetric latin squares and edge-colourings of complete graphs, it is useful to consider embedding of a symmetric latin rectangle R in a symmetric latin square S where *both* entries of S on the diagonal outside R *and* entries of S corresponding to independent edges

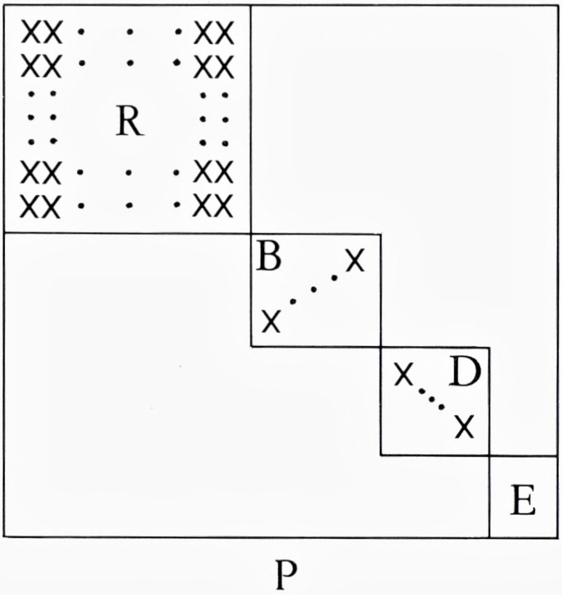


Figure 6

outside the complete subgraph determined by R are prescribed. This can be pictured as in Figure 6.

Theorem 3.8. Let P be a partial symmetric latin square of side n on symbols 1,...,n, and let P be of the form of Figure 6 with the non-empty cells being exactly all cells of R, the back diagonal of B and the diagonal of D. Here R is an $r \times r$ symmetric latin rectangle and B has even side. Some of R, B, D, and E may have side zero, but if B has positive side, then so has E. For all i, $1 \leq i \leq n$, let f(i) be the number of times the symbol i occurs in $B \cup D$.

Then P can be completed to a symmetric latin square of side n if and only if (i) and (ii) hold.

(i) $R(i) \geq 2r-n+f(i)$ for all i, $1 \leq i \leq n$.
(ii) $R(i) + f(i) \equiv n \pmod 2$ for at least $r+d$ different $i \in \{1,\ldots,n\}$, where d is the side of D.

If n is odd, condition (ii) reduces to requiring that all symbols on the diagonal of P are distinct.

Theorem 3.8 extends a known result with B and E not appearing (Hoffman 1983; Andersen 1982). This in turn generalized a wellknown theorem due to A. B. Cruse.

Corollary 3.9. (Cruse 1974). Let R be an $r \times r$ symmetric latin rectangle on $1,\dots,n$. Then R can be embedded in a symmetric latin square of side n if and only if

$R(i) \geq 2r-n$ for all i, $1 \leq i \leq n$, and
$R(i) \equiv n \pmod 2$ for at least r different $i \in \{1,\dots,n\}$.

Proof. Put $f(i) = 0$ for all i, $1 \leq i \leq n$, in Theorem 3.8.

Cruse also obtained the following results. They are easy consequences of Corollary 3.9. All inequalities are best possible.

Corollary 3.10. A partial symmetric latin square of side n on t symbols can be embedded in a symmetric latin square of side t for all even $t \geq 2n$.

Corollary 3.11. A partial symmetric latin square of side n on t symbols, in which each symbol occurs at most once on the diagonal, can be embedded in a symmetric latin square of side t for all $t \geq 2n$.

Corollary 3.12. A partial symmetric idempotent latin square of side n on t symbols can be embedded in a symmetric idempotent latin square for all odd $t \geq 2n+1$.

4. Outline of the proofs of the next sections, and some lemmas

In the next section we shall characterize all partial latin squares of side n with at most n+1 non-empty cells that cannot be completed to a latin square. In Section 6 we do the same for partial symmetric latin squares. We only give a complete proof in the non-symmetric case, but the course of proof is fairly similar for both results. We now give a very broad outline of these proofs, and the remainder of this section will be devoted to some rather technical lemmas to be used.

Both proofs are by induction on n, and the general induction step is as follows. We take the partial square P which is to be completed, delete the entry of one or two cells to obtain a partial latin square P' of smaller side, satisfying our conditions. Then we complete P' by the induction hypothesis. We now partition the completion as in Figure 5 or Figure 6 so that all non-empty cells of P are in R, on the diagonal outside R or, in the symmetric case, on the back diagonal of B of Figure 6. We then introduce the missing symbol or symbols in R in such a way so as to be able to apply Theorem 3.2 or Theorem 3.8 to embed R in a latin square of side n, which will then be a completion of P.

This is a very brief description of the proofs, leaving out a large number of details and of cases not fitting into the general pattern, but hopefully it gives the reader some motivation to try understanding the lemmas of this section.

The first lemma (Andersen and Hilton 1983) will be used in obtaining bounds on the latin rectangle R. A cell of a partial latin square (or of any array) is called *diagonal* if it is the sole non-empty cell in its row and its column. In Figure 6, all cells of $B \cup D$ are diagonal.

Lemma 4.1. Let R be an $r \times s$ array containing exactly q non-empty cells of which none are diagonal, and having no empty row or column. Then

$$r+s \leqslant \left\lfloor \frac{3q}{2} \right\rfloor.$$

Proof. Let r_2 be the number of rows with at least 2 non-empty cells. Then $2r_2 + (r-r_2) \leqslant q$, implying $r \leqslant q - r_2$. Each column of R containing no cell from these r_2 rows must contain at least 2 of the remaining $r - r_2$ cells, and so $s \leqslant \frac{1}{2}(r-r_2) + q - (r-r_2) = q - \frac{1}{2}(r-r_2)$, implying $2s + r \leqslant 2q + r_2$. Therefore $2(r+s) \leqslant q - r_2 + 2q + r_2 = 3q$, as required.

If the non-empty cells are symmetrically placed, and if d of them occur on the diagonal of R, Lemma 4.1 can be strengthened to $2r \leqslant \left\lfloor \frac{3q-d}{2} \right\rfloor$.

The next lemmas will be applied in finding the cells of R whose entries should be changed into a new symbol used in the completion of P, but not in that of P'. In the non-symmetric case, all cells changed in this way must have distinct symbols, and they must not be preassigned in P'. A *partial transversal* of a latin rectangle is a set of cells in distinct rows, in distinct columns, and containing distinct symbols. The *length* of a partial transversal is the number of cells.

Lemma 4.2. Let R be an $x \times y$ latin rectangle and assume that R contains p forbidden cells with at least one in each column, and that

$$(p+1-x)\,(p-y) \geqslant p > 0$$

Let s be one of the symbols of R.

Then R has a partial transversal of length $t \geqslant x+y-p$ avoiding all forbidden cells and all cells containing the symbol s.

Proof. Let R be on the symbols 1, ..., n. From the assumptions $p \geqslant y$, so the inequality gives $p > y$ and $p \geqslant x$. Now $p = x$ would imply $p-y \geqslant p$ which is impossible. So we have $x < p$ and $y < p$. Let t be the length of a maximum partial transversal avoiding the forbidden cells and the symbol s. The result is true if $t \geqslant x - 1$ or $t \geqslant y - 1$, so we now assume that $t \leqslant x - 2$ and $t \leqslant y - 2$. Then $t \leqslant p - 3$.

We consider a partial transversal of length t with the required properties. We can assume that it consists of cells (1,1), ..., (t,t) with entry i in cell (i,i), $1 \leqslant i \leqslant t$ (so that $s > t$). By the maximality, no symbol from $\{t+1, ..., n\} \setminus \{s\}$ can occur in rows $t + 1, ..., x$ of columns $t+1, ..., y$ except in forbidden cells.

Let

$$A_0 = \emptyset$$
$$A_j = \{ i \in \{1, ..., x\} \mid \text{cell } (i, t+j) \text{ is not forbidden and its entry} \notin (A_{j-1} \cup \{t+1, ..., n\}) \setminus \{s\}\},\ 1 \leqslant j \leqslant y-t.$$

Define an oriented graph $\vec{G}$ on vertices

$$\bigcup_{j=1}^{y-t} (A_j \times \{t+j\})$$

(corresponding to some cells in the last y-t columns) and edges

$$\{((a,t+j), (b,t+k)) \mid j < k \text{ and cell } (b,t+k) \text{ contains the symbol } a\}.$$

We claim:

(⋆) For all j, $1 \leqslant j \leqslant y - t : \{t+1, ..., x\} \cap A_j = \emptyset$.

Proof of (⋆): Suppose that (⋆) fails. Then $\vec{G}$ contains a vertex $(a_j, t+j)$ with $a_j \in \{t+1, ..., x\}$ and $1 \leqslant j \leqslant y - t$. It follows from the definitions that $\vec{G}$ contains a directed path ending in $(a_j, t+j)$ and starting in a vertex $(a_k, t+k)$ with $k < j$, where cell $(a_k, t+k)$ has an entry from $\{t+1, ..., n\} \setminus \{s\}$.

Let $(g_0, t+i_0), (g_1, t+i_1), ..., (g_\ell, t+i_\ell)$ be a shortest directed path of $\vec{G}$ with the property that the entry of cell $(g_0, t+i_0)$ is in $\{t+1, ..., n\} \setminus \{s\}$ and $g_\ell \in \{t+1, ..., x\}$. So $g_k \leqslant t$ for $0 \leqslant k \leqslant \ell -1$.

Then the cells

$(g_k, t+i_k)$, $0 \leqslant k \leqslant \ell$, and
(j, j), $1 \leqslant j \leqslant t$, $j \neq g_k$ for $0 \leqslant k \leqslant \ell-1$,

form a partial transversal of length t+1, contradicting the maximality of t. We prove that these t+1 cells do indeed form a partial transversal satisfying the requirements:

(i) By definition, the cells are in R, they are not forbidden and their entries are different from s.

(ii) The cells are in distinct rows, because otherwise we would have $g_h = g_k$ for some $h < k$, where $k \neq \ell$ as $g_h \leqslant t < g_\ell$, and then the entry of cell $(g_{k+1}, t+i_{k+1})$ would be $g_k = g_h$, so that $(g_0, t+i_0), \ldots, (g_h, t+i_h), (g_{k+1}, t+i_{k+1}), \ldots, (g_\ell, t+i_\ell)$ would be a shorter path.

(iii) The cells are obviously in distinct columns.

(iv) The entries are distinct, namely $\{1, \ldots, t\}$ and the entry of $(g_0, t+i_0)$. This completes the proof of (*).

Let p_j be the number of forbidden cells and δ_j the number of occurrences of the symbol s in the column j, for $1 \leqslant j \leqslant y$. Then $p_j \geqslant 1$ by assumption, and $\delta_j \in \{0,1\}$.

From the definition of A_j and from (*) we get

$$|A_j| \geqslant x - (t-|A_{j-1}| + \delta_{t+j}+p_{t+j})$$

giving

$$|A_{y-t}| \geqslant (y-t)(x-t) - \sum_{j=1}^{y-t} \delta_{t+j} - \sum_{j=1}^{y-t} p_{t-j}.$$

By (*), $|A_{y-t}| \leqslant t$, and as $\sum_{j=1}^{y-t} p_{t+j} = p - \sum_{j=1}^{t} p_j \leqslant p - t$

and $\sum_{j=1}^{y-t} \delta_{t+j} \leqslant y-t$ we get

$$t \geqslant (y-t)(x-t) - (y-t) - (p-t)$$

and, introducing the condition of the lemma,

$$\begin{aligned} 0 &\leqslant p - (y-t)(x-t-1) \\ &\leqslant (p+1-x)(p-y) - (y-t)(x-t-1) \\ &= (t-x-y+p+1)(p-t). \end{aligned}$$

If we have strict inequality somewhere in these calculations, we get $t-x-y+p+1 > 0$ (as $p-t \geqslant 3$) implying $t \geqslant x+y-p$ as required. So assume that $t = x+y-p-1$ and that we have equality; in particular $p_j = 1$ for $1 \leqslant j \leqslant t$, $(y-t)(x-t-1) = p$ and $|A_{y-t}| = t$, from which we deduce that column y has no forbidden cell and no cell containing s among the first t cells. By the maximality of t, this must be true with any ordering of the last y-t columns, so in fact it holds for each of columns t+1, …, y. We now prove

(★★) Each entry in any of the first t cells of any of the last y-t columns belongs to $\{1, ..., t\}$.

Proof of (★★):
As $p > y$ we may assume that $p_k \geqslant 2$ for some k, $t + 1 \leqslant k \leqslant y$. If some cell (i,k), $1 \leqslant i \leqslant t$, contains a symbol $u > t$, we can replace (i,i) of our partial transversal with (i,k), and we no longer have $p_j = 1$ for all columns j containing a cell from the transversal. Assume next that some cell (i,ℓ) contains a symbol $u > t$, where $1 \leqslant i \leqslant t$ and $t + 1 \leqslant \ell \leqslant y$, $\ell \neq k$. Some cell of column k contains the symbol i, say cell (j,k). Then $1 \leqslant j \leqslant t$, $j \neq i$, and replacing cells (i,i) and (j,j) by (i,ℓ) and (j,k) we obtain the same contradiction as before. This proves (★★).

It follows from (★★) that the cells common to the last $x - t$ rows and the last $y - t$ columns contain symbols greater than t. Any such cell can be added to the partial transversal unless it is forbidden or contains s. So we get

$$(x-t)\ (y-t) \leqslant (p-t) + (y-t)$$

contradicting $(y-t)\ (x-t-1) = p$, and so Lemma 4.2 has been proved.

Lemma 4.2 strengthens a result due to A. J. W. Hilton and the author (Andersen & Hilton 1983). The similar lemma for the symmetric case is most easily stated in graph terminology. A *path system* of a graph is a subgraph consisting of disjoint paths. As we only aim at sketching the proof in the symmetric case we state the lemma in a form less complicated than what is needed in the proof.

Lemma 4.3. Let $4 \leqslant r \leqslant \frac{3n-6}{4}$ and let K_r have an edge-colouring with any number of colours. Let F be a set of at most $\frac{n+1}{2}$ forbidden edges of K_r such that each vertex is incident with an edge of F. Let M be a set of at most 2 mandatory edges of K_r, disjoint from F and not containing 2 edges of the same colour.

Then K_r contains a path system containing all edges from M and no edge from F, with all edges having distinct colours, and with at least 2r–n edges.

The proofs of Lemmas 4.2 and 4.3 have been inspired by work on the existence of long partial transversals in latin squares (Drake 1977; Brouwer, de Vries & Wieringa 1978; Woolbright 1978). It can be proved that an edge-coloured K_n has a path system with all edges having distinct colours with at least $n - \sqrt{2n}$ edges (Andersen 1985).

5. Completion of partial Latin Squares

This section is primarily devoted to characterizing those partial latin squares of side n with at most n+1 non-empty cells which cannot be completed to a latin square of side n, thus extending the knowledge gained from previous proofs of the Evans Conjecture.

A forerunner for the complete proofs was a paper by R. Häggkvist, where he proved that the conjecture is true for $n > 1111$ (Häggkvist 1978); we shall use one of Häggkvist's lemmas in our proof. The proof of B. Smetaniuk was based on a remarkable completion theorem, which we state below (although we shall not apply the result here).

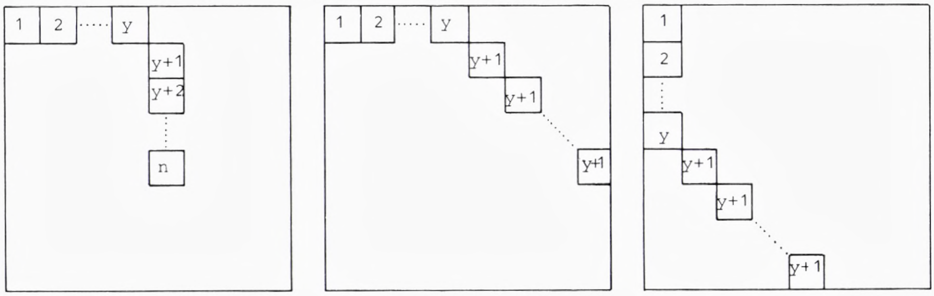
Theorem 5.1. (Smetaniuk 1981). Let A be any latin square of side n on symbols 1,...,n, and let P(A) be the partial latin square of side n+1 on symbols 1,...,n+1 in which, for all i,j, $1 \leqslant i \leqslant n+1$, $1 \leqslant j \leqslant n+1$, cell (i,j) contains the entry of cell (i,j) of A if (i,j) is above the back diagonal of P(A) (so that $1 \leqslant j < n+2-i$), cell (i,j) contains the symbol n+1 if (i,j) is on the back diagonal of P(A) ($j=n+2-i$), and otherwise cell (i,j) is empty.

Then P(A) can be completed to a latin square of side n+1.

Smetaniuk actually gave a specific algorithm for completing P(A), and he showed that if $A \neq B$, then the completions of P(A) and P(B) obtained in this way are also different. It follows that the number of latin squares of side n is a strictly increasing function of n (Smetaniuk 1982).

The proof of the Evans Conjecture by A. J. W. Hilton and the author also proved that a partial latin square of side n with exactly n non-empty cells can be completed unless it is of the form of one of the partial squares of Figure 7, $1 \leqslant y \leqslant n-1$, (i.e., by permuting the rows, permuting the columns and renaming the symbols it can be transformed into one of these squares). This was actually conjectured to be true by D. Klarner in 1970 in a conversation with Hilton. In 1983 R. M. Damerell showed that it can be proved using Theorem 5.1 (Damerell 1983).

The theorem that we shall prove in this section states that if a partial latin square of side n with n+1 non-empty cells cannot be completed then it is of the form of one of the squares of Figure 8, or n=4 and it is as in Figure 9, or it contains one of the squares of Figure 7.



Type 1 ($n \geq 2$) Type 2 ($n \geq 2$) Type 3 ($n \geq 2$)

Fig. 7

We call a partial latin square of side n with at most n+1 non-empty cells *bad* if it has n or n+1 non-empty cells of which n cells form a square of the type of one of the squares of Figure 7, $1 \leqslant y \leqslant n-1$, or it has n+1 non-empty cells forming a square of the type of one of the squares of Figure 8 or Figure 9; otherwise we call it *good*. If a good partial latin square of side $n \geqslant 2$ has less than n+1 non-empty cells, we can fill further cells so as to obtain a good square with exactly n+1 non-empty cells.

It is easy to see that a bad partial latin square of side n cannot be completed to a latin square of side n; we leave this little exercise to the reader.

In section 2, it was explained how a latin square of side n corresponds to a decomposition of $K_{n,n,n}$ into mutually edge-disjoint K_3's. It follows from this that there is symmetry among rows, columns and symbols. For example, if S is a latin square of side n and S' is obtained from S by placing the symbol j in cell (i,k) whenever S contains the symbol k in cell (i,j), $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$, then S' is also a latin square of side n. We say that S' is obtained from S by *interchanging columns and symbols*. Similarly, other permutations of (rows, columns, symbols) give rise to latin squares. We call these *conjugates* of S. Conjugates of partial latin squares are defined in the same way.

Clearly, a partial latin square of side n can be completed to a latin square of side n if and only if any one of its conjugates can.

In Figures 7 and 8, partial latin squares in the same row are conjugates of each other. All conjugates of the square of Figure 9 are of the same form.

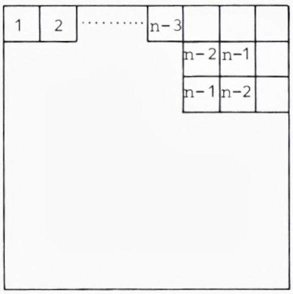
The proof in this section is very similar to that of A. J. W. Hilton and the author for the case of n non-empty cells. Some proofs are almost identical, others are a bit more complicated in this paper.

We first verify the result in a particular case where the general proof does not work.

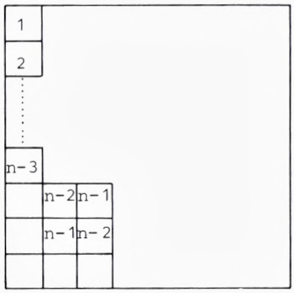
Fig. 9

1	2		
	1		
		3	
			4

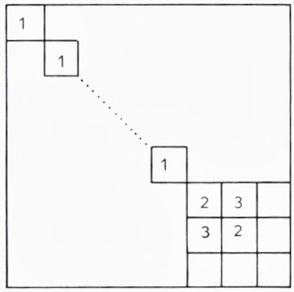
Fig. 8



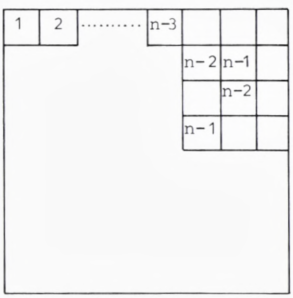
Type 4 ($n \geq 3$)



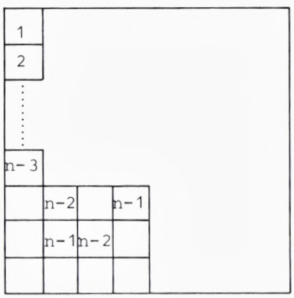
Type 5 ($n \geq 3$)



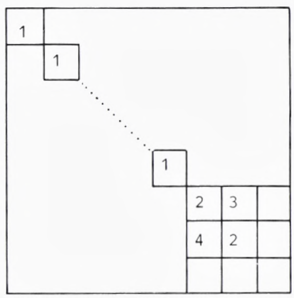
Type 6 ($n \geq 3$)



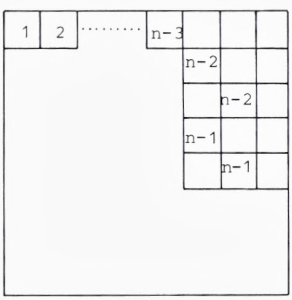
Type 7 ($n \geq 4$)



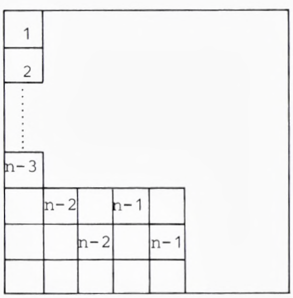
Type 8 ($n \geq 4$)



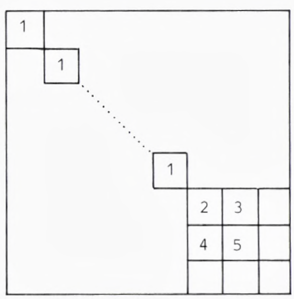
Type 9 ($n \geq 4$)



Type 10 ($n \geq 5$)

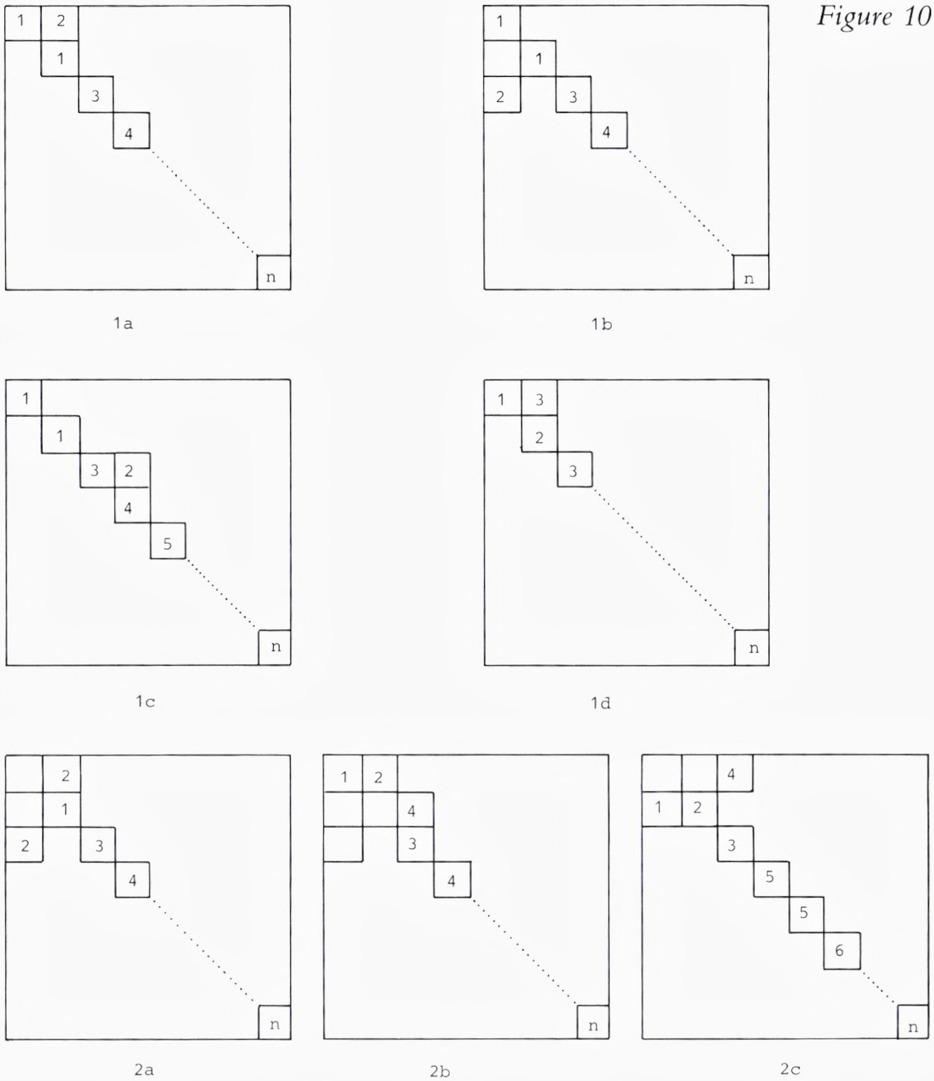


Type 11 ($n \geq 5$)



Type 12 ($n \geq 5$)

Figure 10



Lemma 5.2. If P is a good partial latin square of side n with n+1 non-empty cells such that each row and each column contains a non-empty cell, and each of symbols 1,...,n occurs in P, then P can be completed to a latin square of side n.

Proof. If we pick n non-empty cells of P belonging to distinct rows, then at most 2 of them can belong to the same column. It follows that P contains at least n−1 non-empty cells belonging to distinct rows and distinct columns. We can distinguish between two cases, according to whether n such cells exist or not. Considering the different positions for the unique symbol occurring twice, we see that P must be of the form of one of the partial latin squares of Figure 10.

We prove cases 1a, 1b and 2a simultaneously. First for n⩾11. By Theorem 3.7 there is an idempotent latin square S_5 of side 5 with diagonal 1,2,3,4,5, and by the same theorem there is an idempotent latin square of side n on symbols 1,2,...,n with S_5 in the top left hand corner. If we replace this S_5 by the latin square of side 5 in Figure 11, we still have a latin square, and it clearly is a completion of the squares of cases 1a, 1b, and 2a. For 5⩽n⩽10 completions of all 3 cases are shown in Figure 11. For n=4 case 1a gives a bad square, and cases 1b and 2a are easily completed. For n=3 all three cases give bad squares (and for n=2 only case 1a applies and is trivial).

Case 1c also follows from the above constructions (for all n⩾5, the symbol 2 not on the diagonal can be found in the row of the diagonal 4, in column 5 or 6). For n=4, case 1c gives a bad square.

Case 1d is obvious, it follows from the existence of idempotent latin squares of side n for all n⩾3.

Case 2b yet again follows from Figure 11 and the construction related to it (for 6⩽n⩽10, one of symbols 5 and 6 is repeated rather than the symbol 4). For n=4, case 2b gives a bad square.

Case 2c only applies for n⩾5, and here we copy the argument for the first three cases, but with the latin squares of Figure 12.

1	2	5	3	4
5	1	4	2	3
2	4	3	5	1
3	5	1	4	2
4	3	2	1	5

1	2	6	5	4	3
3	1	5	6	2	4
2	4	3	1	6	5
5	6	1	4	3	2
6	3	4	2	5	1
4	5	2	3	1	6

1	2	5	7	4	3	6
3	1	6	5	7	4	2
2	4	3	1	6	7	5
5	6	7	4	3	2	1
6	7	4	2	5	1	3
7	5	2	3	1	6	4
4	3	1	6	2	5	7

Figure 11

1	2	5	7	8	9	6	4	0	3
3	1	6	8	7	4	9	0	2	5
2	4	3	1	6	0	5	9	7	8
9	6	0	4	3	2	8	1	5	7
8	7	9	0	5	1	3	6	4	2
7	5	8	3	9	6	0	2	1	4
4	0	2	6	1	5	7	3	8	9
6	9	4	5	0	7	2	8	3	1
0	8	7	2	4	3	1	5	9	6
5	3	1	9	2	8	4	7	6	0

1	2	5	7	8	9	6	4	3
3	1	6	8	7	4	9	5	2
2	4	3	1	6	8	5	9	7
9	6	7	4	3	2	8	1	5
8	7	9	2	5	1	3	6	4
7	5	8	3	9	6	4	2	1
4	9	2	6	1	5	7	3	8
5	3	4	9	2	7	1	8	6
6	8	1	5	4	3	2	7	9

1	2	5	7	8	3	6	4
3	1	6	8	7	4	2	5
2	4	3	1	6	8	5	7
5	6	7	4	3	2	8	1
8	7	4	2	5	1	3	6
7	5	8	3	1	6	4	2
4	8	1	6	2	5	7	3
6	3	2	5	4	7	1	8

5	3	4	1	2
1	2	5	4	3
4	5	3	2	1
3	1	2	5	4
2	4	1	3	5

2	6	4	3	1	5
1	2	5	6	4	3
5	1	3	2	6	4
3	4	6	5	2	1
6	3	1	4	5	2
4	5	2	1	3	6

2	6	4	3	7	5	1
1	2	7	6	4	3	5
5	1	3	2	6	7	4
7	4	6	5	2	1	3
3	7	2	1	5	4	6
4	3	5	7	1	6	2
6	5	1	4	3	2	7

2	6	4	9	0	5	8	1	3	7
1	2	7	6	9	0	5	3	4	8
5	1	3	2	8	7	0	9	6	4
7	8	6	5	2	9	3	4	0	1
3	9	8	1	5	4	6	0	7	2
4	5	0	8	3	6	1	7	2	9
9	0	1	4	6	2	7	5	8	3
6	4	2	0	7	3	9	8	1	5
0	3	5	7	1	8	4	2	9	6
8	7	9	3	4	1	2	6	5	0

2	6	4	9	7	5	8	1	3
1	2	7	6	9	8	5	3	4
5	1	3	2	8	7	4	9	6
7	8	6	5	2	9	3	4	1
3	9	8	1	5	4	6	2	7
4	5	9	8	3	6	1	7	2
9	3	1	4	6	2	7	5	8
6	7	2	3	4	1	9	8	5
8	4	5	7	1	3	2	6	9

2	6	4	3	7	5	8	1
1	2	7	6	4	8	5	3
5	1	3	2	8	7	4	6
7	8	6	5	2	1	3	4
3	7	8	1	5	4	6	2
4	5	2	8	3	6	1	7
8	3	1	4	6	2	7	5
6	4	5	7	1	3	2	8

Figure 12

This completes the proof of Lemma 5.2.

Lemma 5.4 below is a strengthening of a very useful result due to C. C. Lindner, which was used also by Smetaniuk and by Damerell (Lindner 1970). We first state another lemma; a *1-factor* of a graph G is the edge-set of a subgraph F with the same vertex set as G and with each vertex having degree 1 (sometimes the term 1-factor is used for the subgraph itself and not just the edge-set).

Lemma 5.3. (Häggkvist 1978). Let G be a regular bipartite graph of degree m with 2n vertices. Let B_1 be a set of b_1 independent edges, and let B_2 be a set of b_2 edges disjoint from B_1. If $m\text{-}b_1 \geqslant \frac{1}{2}(n-1)$ and $b_1+b_2 \leqslant m-1$, then G contains a 1-factor F such that $B_1 \subseteq F$ and $F \cap B_2 = \emptyset$.

Lemma 5.4. Let P be a good partial latin square of side n with exactly $n+1$ non-empty cells. Let the number of non-empty cells in row i be r_i, $1 \leqslant i \leqslant n$, and assume that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n = 0$. Then the first $\lfloor \frac{1}{2}(n+1) \rfloor$ rows of P can be completed.

Proof. The particular case where $n=4$, $r_1=r_2=2$ and $r_3=1$ turns out to be an exception in several arguments. Rather than go through all the details every time the exception is encountered, we ask the reader to verify the lemma in this case. So we shall assume that if $n=4$ then $r_1 \neq 2$.

With P we associate a bipartite graph $K_{n,n}$ with vertex classes C and S corresponding to columns and symbols respectively. For $1 \leq i \leq n$, let B_i be the set of r_i independent edges of $K_{n,n}$ corresponding to the non-empty cells of row i, i.e. the edge joining column j and symbol k is in B_i if and only if k is in cell (i,j) of P ($1 \leq j \leq n$, $1 \leq k \leq n$). Extending B_i to a 1-factor corresponds to assigning a symbol to each cell of row i.

We first prove that we can complete the first row of P. Let $G=K_{n-r_1,n-r_1}$ be obtained from $K_{n,n}$ by deleting all end-vertices of edges of B_1. We must find a 1-factor of G, disjoint from the set $B=(B_2 \cup B_3 \cup \ldots \cup B_n) \cap E(G)$ of at most $n+1-r_1$ edges. By a wellknown theorem (Hall 1935), it suffices to show that in G-B any k vertices from C have at least k neighbours in S altogether, $1 \leq k \leq n-r_1$. This is true for $k=n-r_1$ because otherwise B would contain $n-r_1$ edges of G incident with the same vertex of S, and so P would be a bad square, containing a Type 2 square. If it fails for $k=n-r_1-1$, then B must contain all edges between 2 vertices of S and the k vertices of C, and so $2(n-r_1-1) \leq n+1-r_1$ implying $n-r_1 \leq 3$; if $n-r_1=3$ it implies that P is of Type 4, 7 or 10, and if $n-r_1=2$ it implies that P contains a Type 1 square with $y=n-2$, both cases contradicting that P is good. Hall's condition cannot fail for a k with $3 \leq k \leq n-r_1-2$, because then B would contain at least $k(n-r_1-k+1) > n-r_1+1$ edges. If it fails for $k=2$ we would get $n-r_1 \leq 3$ as above, and so $k=n-r_1$ or $k=n-r_1-1$, both cases covered above. Finally, if it fails for $k=1$ then P contains a Type 1 square, which is a contradiction. Thus we have proved that the first row can be completed.

Now suppose that we have a sequence of graphs $G_0, G_1, \ldots, G_p$, where $G_0=K_{n,n}$ and, for $1 \leq r \leq p$, $G_r=G_{r-1}-F_r$, where F_r is a 1-factor of G_{r-1} containing B_r and disjoint from $B_{r+1}, \ldots, B_n$. This corresponds to p rows having been completed. The sequence exists for $p=1$. We assume that $p < \lfloor \frac{1}{2}(n+1) \rfloor$ and want to extend the sequence by finding a 1-factor F_{p+1} of G_p containing B_{p+1} and disjoint from $B_{p+2}, \ldots, B_n$. In most cases, this can be done by applying Lemma 5.3; some cases are done separately.

Let $G=G_p$, $b_1=r_{p+1}$, $b_2 = \sum_{i=p+2}^{n} r_i$ and $m=n-p$. We examine the two inequalities of Lemma 5.3 one by one.

We must have $r_1 \geq 2$ and so

$$b_1+b_2 = \sum_{i=p+1}^{n} r_i = n+1-\sum_{i=1}^{p} r_i \leq n+1-(p+1) = n-p = m$$

so $b_1+b_2 \leq m-1$ if we have strict inequality. If $p \geq 2$ then there is strict inequality, because $r_1+r_2 \geq 4$ as $r_n=0$. If $r_1 \geq 3$ the strict inequality is also satisfied, so we now consider the case $p=1$, $r_1=2$ separately. Then $r_2=2$, and, by assumption, $n \neq 4$. We use the same method as we did when completing the first row. Let G' be obtained from G by deleting the end-vertices of the edges of B_2, and all edges of $B=(B_3 \cup \ldots \cup B_n)$. Then G' is $K_{n-2,n-2}$ with a set of at most $n-2$ independent edges (from F_1) and a set of at most $n-3$ further edges deleted. We use Hall's condition to find a 1-factor of G', considering k vertices of C, $1 \leqslant k \leqslant n-2$. If it fails for $k=n-2$, then all edges incident with some vertex of S have been deleted, corresponding to having a partial latin square as in Figure 13. But then the symbol 3 in the first row is not preassigned, because if it were, P would contain a Type 2 square with $y=2$. It follows that we can change the first row so as to place 3 elsewhere, as P is not of Type 6, 9 or 12. If the Hall condition fails for $k=n-3$ then at least $2(n-3)$ edges incident with 2 particular vertices have been deleted, and as at most 2 of these can be in the set of independent edges we get $2(n-3) \leqslant 2+(n-3)$ implying $n-2 \leqslant 3$. It follows that we have one of the situations of Figure 14. Then we can change the first row so as to have the condition satisfied for $k=2$, as P is not of Type 4 or 7 (and it will still be satisfied for $k=3=n-2$, as Figure 13 does not apply). If $3 \leqslant k \leqslant n-4$ the Hall condition cannot fail, because then $n \geqslant 7$ and at least $k(n-1-k) - \min\{k, n-1-k\}$ edges not among the independent edges have been deleted, which implies $n \leqslant 6$. If the condition fails for $k=2$ we can deduce $n \leqslant 5$ and so $k=n-3$. If it fails for $k=1$ then $n \geqslant 5$ as the case $n=3$ and $k=n-2$ is covered above, and we have the situation of Figure 15. Unless the symbol 3 is prescribed in its cell, or both symbols 1 and 2 are prescribed somewhere in row 1, we can interchange occurrences of 3 and either 1 or 2 to avoid the situation; but in these cases P is bad (Type 1 with $y=2$, Type 5, 8 or 11).

So henceforth, when trying to find the 1-factor F_{p+1}, we can assume that the inequality $b_1+b_2 \leqslant m-1$ holds.

We now consider the other inequality of Lemma 5.3, which in our case is $r_{p+1} \leqslant \frac{1}{2}(n+1)-p$. As $p < \lfloor \frac{1}{2}(n+1) \rfloor$ this is true if $r_{p+1}=1$. So assume that $r_{p+1} \geqslant 2$. If $r_p \geqslant r_{p+1}+1$ then $r_1 \geqslant 3$ and we get, for $p \geqslant 2$,

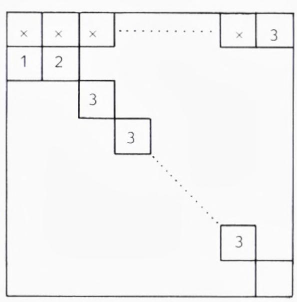


Fig. 13

×	×	3	4	×
1	2			
		4	3	

×	×	3	4	×
1	2			
		4		
			3	

Fig. 14

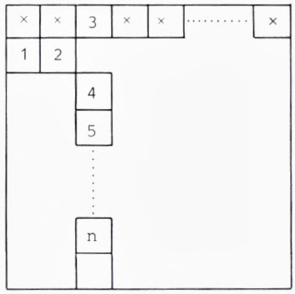


Fig. 15

$$r_{p+1}=r_2\leqslant\tfrac12(r_1+r_2-1)\leqslant\tfrac12\Big(n-\sum_{i=3}^{n} r_i\Big)\leqslant\tfrac{n}{2}=\tfrac12(n+1)-p+\tfrac12$$

and for $p=1$:

$$r_{p+1}=r_2\leqslant\tfrac12(r_1+r_2-1)\leqslant\tfrac12\Big(n-\sum_{i=3}^{n} r_i\Big)\leqslant\tfrac{n}{2}n=\tfrac12(n+1)-p+\tfrac12$$

so the inequality is true unless $r_3=r_4=\ldots=r_n=0$ and n is even, $r_1=\frac{n}{2}+1$ and $r_2=\frac{n}{2}$. But in that case we can find a 1-factor directly: delete all end-vertices of edges of B_2 from G_1; we must find a 1-factor in the remaining graph, which is just $K_{\frac{n}{2},\frac{n}{2}}$ with some or all edges of a 1-factor deleted, and this can be done by Hall's condition, because if $k\geqslant2$ then all vertices in the other class is joined to one of the k vertices, and a single vertex ($k=1$) has a neighbour, because $\frac{n}{2}>1$.

So we now assume that $r_p=r_{p+1}$. If $\sum_{i=p+2}^{n} r_i\geqslant2$ then

$$r_{p+1}=\tfrac12(r_p+r_{p+1})\leqslant\tfrac12\Big(n+1-\sum_{i=1}^{p-1} r_i-\sum_{i=p+2}^{n} r_i\Big)\leqslant\tfrac12(n+1-2(p-1)-2)$$

and the inequality is satisfied. Now assume that $\sum_{i=p+2}^{n} r_i=1$. Then the inequality is true if $\sum_{i=1}^{p-1} r_i > 2(p-1)$, and if this is not the case, then *either* $p=1$ and so $r_p=r_{p+1}=\frac{n}{2}$, *or* $r_1=r_2=\ldots=r_{p+1}=2$ implying $p=\frac{n}{2}-1$. In both cases n is even and, by assumption, $n\geqslant6$. In the former case the usual method works without problems; we now consider the latter case. In G_p, each vertex has degree $\frac{n}{2}+1$. Let G' be obtained from G by deleting the end-vertices of the two edges of B_{p+1} and, if neither of its end-vertices have been deleted, the single edge e of B_{p+2}. Then each vertex has degree at least $\frac{n}{2}-1$ in G' except possibly the end-vertices of e which may have degree $\frac{n}{2}-2$. Hence Hall's condition is certainly satisfied for $k\leqslant\frac{n}{2}-2$ and also for $k=\frac{n}{2}-1$ because $\frac{n}{2}-1>1$ so there is a vertex not incident with e in any set of k vertices. If $k\geqslant\frac{n}{2}$ then any vertex in the other class will be joined to at least one of the k vertices (except possibly an end-vertex of e), and the condition holds.

The only case left to consider in trying to establish the inequality $r_{p+1} \leqslant \frac{1}{2}(n+1)-p$ is when $r_{p+2}=r_{p+3}=\ldots=r_n=0$. From before we have that $r_p=r_{p+1} \geqslant 2$. We show that the inequality can fail in four ways. (i) If $p=1$ then n is odd and $r_1=r_2=\frac{1}{2}(n+1), r_3=r_4=\ldots=r_n=0$. If $p \geqslant 2$ the inequality is true for $r_1 \geqslant 4$, as

$$r_{p+1}=\tfrac{1}{2}(r_p+r_{p+1}) = \tfrac{1}{2}(n+1-\sum_{i=1}^{p-1} r_i) \leqslant \tfrac{1}{2}(n+1-r_1-2(p-2)) \leqslant \tfrac{1}{2}(n+1)-p,$$

so we can assume that $r_1 \leqslant 3$. If $r_1=r_2=3$ then it is satisfied for $p \geqslant 3$; but we get the exception (ii) $p=2$, $n=8$, $r_1=r_2=r_3=3$, $r_4=r_5=\ldots=r_8=0$. Finally, we have exceptions (iii) $r_1=3$, $r_2=r_3=\ldots=r_{p+1}=2$, $r_{p+2}=r_{p+3}=\ldots=r_n=0$, $p=\frac{n}{2}-1$, n even and $n \geqslant 6$, and (iv) $r_1=r_2=\ldots=r_{p+1}=2$, $r_{p+2}=r_{p+3}=\ldots=r_n=0$, $p=\frac{1}{2}(n-1)$, n odd and $n \geqslant 5$. In each exceptional case we apply Hall's condition on the usual subgraph of $K_{n-r_{p+1},n-r_{p+1}}$ to try to find a 1-factor.

In case (i) we have $K_{\frac{1}{2}(n-1),\frac{1}{2}(n-1)}$ with some independent edges deleted. Hall's condition cannot fail unless $\frac{1}{2}(n-1)=1$, so that $n=3$, and if it does then P is easily seen to be bad. In case (ii), we have a subgraph of $K_{5,5}$ in which each vertex has degree at least 3, and Hall's condition is easily seen to be satisfied. Case (iii) gives us a subgraph of $K_{n-2,n-2}$ in which each vertex has degree at least $n-2-(\frac{n}{2}-1)=\frac{n}{2}-1$; but then Hall's condition is obviously true for $k \leqslant \frac{n}{2}-1$, and for $k \geqslant \frac{n}{2}$ it is true because any vertex in the other class must be joined to one of the k vertices.

Finally, in case (iv) we are looking at a subgraph of $K_{n-2,n-2}$ in which each vertex has degree at least $n-2-\frac{1}{2}(n-1)=\frac{1}{2}(n-1)-1$. So Hall's condition is true for $k \leqslant \frac{1}{2}(n-1)-1$. It is also true for $k \geqslant \frac{1}{2}(n-1)+1$, because in that case each vertex in the other class is joined to one of the k vertices. However, the condition may fail for $k=\frac{1}{2}(n-1)$. If it does, we can describe the graph G_p (in which every vertex has degree $\frac{1}{2}(n+1)$) very accurately: Let the edges of B_{p+1} have end-vertices c_1 and c_2 in C, s_1 and s_2 in S. Then $C=\{c_1,c_2\} \cup A \cup B$ and $S=\{s_1,s_2\} \cup T \cup U$, where $|A|=|U|=\frac{1}{2}(n-1)$, $|B|=|T|=\frac{1}{2}(n-1)-1$, every vertex of A is joined to every vertex of $T \cup \{s_1,s_2\}$, every vertex of U is joined to every vertex of $B \cup \{c_1,c_2\}$ and apart from these edges, G_p contains c_1s_1, c_2s_2 and $\frac{1}{2}(n-1)-1$ independent edges each joining a vertex from B to a vertex from T. Figure 16 illustrates the graph G_p and the corresponding partial latin square.

It follows from the structure of the graph that the p rows completed so far actually have p of the columns forming a latin square. The remaining columns form a latin square with one row missing. We can then simply find a row which contains a non-preassigned entry from each latin

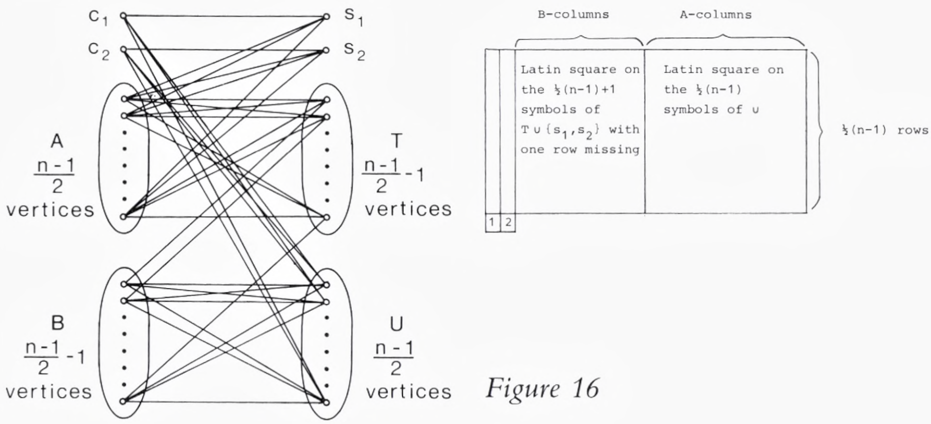


Figure 16

square and interchange the two entries. Then we no longer have the situation of Figure 16, and hence we can complete the (p+1)st row. If no row contains two such non-preassigned entries, then $\frac{1}{2}(n-1)=2$ implying $n=5$, and all cells in the 2×2 latin square are preassigned; but then P is of type 4, which is a contradiction.

We have now shown how to complete row p+1 in all cases not satisfying the conditions of Lemma 5.3. If those conditions are satisfied, the lemma provides us with the required 1-factor, thus enabling us to fill row p+1. This completes the proof of Lemma 5.4.

Corollary 5.5. If P is a good partial latin square of side n with n+1 non-empty cells, and if P has an empty row and all non-empty cells outside a given column lie in $\lfloor \frac{1}{2}(n+1) \rfloor$ rows, then P can be completed to a latin square of side n.

Proof. By Lemma 5.4, the rows containing the non-empty cells outside the given column can be completed, and by Corollary 3.5 the partial latin square containing these rows can be completed; but then it is just a matter of permuting the remaining rows to get the column right.

Corollary 5.6. A good partial latin square of side n with n+1 non-empty cells, all lying in $\lfloor \frac{1}{2}(n+3) \rfloor$ rows, can be completed to a latin square of side n.

Lemma 5.7. Let P be a good partial latin square of side n with n+1 non-empty cells all lying in the top left $r\times s$ subarray R or on the diagonal outside R, and assume that $s \leq \lfloor \frac{1}{2}(n+1) \rfloor$, and that P has an empty column.

Then P can be completed to a latin square of side n.

Proof. Figure 5 shows the partition of P. We can assume that R has no empty columns. By Lemma 5.4 with rows replaced by columns, the first s columns can be completed. By the same argument as in the proof of Lemma 3.1 this implies that $R(i) \geq r+s-n+f(i)$ for all i, $1 \leq i \leq n$, where P is supposed to be on symbols $1,2,\ldots,n$ and f(i) is the number of times that symbol i occurs on the diagonal outside R, $1 \leq i \leq n$. But then R can be completed by Theorem 3.2.

Lemma 5.8. Let P be a good partial latin square of side n with n+1 non-empty cells. If each row contains a non-empty cell, then P can be completed to a latin square of side n.

Proof. By Lemma 5.2, it suffices to consider the case where either P has an empty column, or some symbol does not occur in P. We can interchange columns and symbols if necessary, so assume that P has an empty column. Let t be the number of non-empty columns of P, and let v be the number of columns with exactly one non-empty cell. By Corollary 5.6, we can complete P if $t \leq \lfloor \frac{1}{2}(n+3) \rfloor$, so we suppose that $t \geq \lfloor \frac{1}{2}(n+3) \rfloor + 1$. Then

$$t-v=(2(t-v)+v)-t \leq n+1-t \leq n-\lfloor \tfrac{1}{2}(n+3) \rfloor = \lfloor \tfrac{1}{2}(n-2) \rfloor .$$

All the non-empty cells outside the t-v columns are in distinct columns, and they are also in distinct rows except that one row may contain two of them. So all non-empty cells outside s columns are diagonal, where

$$s \leq t-v+2 \leq \lfloor \tfrac{1}{2}(n-2) \rfloor + 2 = \lfloor \tfrac{1}{2}(n+2) \rfloor .$$

So $s \leq \lfloor \frac{1}{2}(n+1) \rfloor$ unless n is even and we have equality everywhere. But in that particular case $t=\lfloor \frac{1}{2}(n+3) \rfloor + 1 = \frac{1}{2}(n+4)$ and $2(t-v)+v=n+1$, implying $v=2t-n-1=3$, and two of the three non-empty cells outside the t-v columns are in the same row; therefore all non-empty cells not in this row are in $\frac{1}{2}(n+4)-2=\frac{n}{2}$ columns, and so P can be completed by Corollary 5.5.

Hence we can assume that $s \leq \lfloor \frac{1}{2}(n+1) \rfloor$. And then P can be completed by Lemma 5.7, and Lemma 5.8 has been proved.

Lemma 5.8 implies that P can be completed if any of its conjugates satisfy the condition. So if P is good and all rows are used, all columns are used, or all symbols are used, then P can be completed.

Corollary 5.9. If P is a good partial latin square of side $n \leqslant 5$ with $n+1$ non-empty cells, then P can be completed.

Proof. For $n \leqslant 5$, $\lfloor \frac{1}{2}(n+3) \rfloor \geqslant n-1$.

We need just one more lemma, before we can prove our main result.

Lemma 5.10. Let P be a partial latin square of side n with q non-empty cells, and with the property that neither P nor any of its conjugates have any diagonal non-empty cells. Let r and s be the number of non-empty rows and columns respectively, and let t be the number of distinct symbols occurring in P. Then

$$\min\{r+s, s+t, t+r\} \leqslant \frac{4q}{3}.$$

Proof. Let R and S be the set of cells in rows with at least two non-empty cells and the set of cells in columns with at least two non-empty cells respectively, and let T be the set of cells containing symbols occurring at least twice in P.

Put $x = |(R \cap T) \setminus S|$, $y = |R \cap S) \setminus T|$, $z = |(T \cap S) \setminus R|$ and $w = |R \cap S \cap T|$.

Then $q = x+y+z+w$, $|R| = x+y+w$ and so

$r \leqslant z + \frac{1}{2}|R| \leqslant \frac{1}{2}(q+z)$

Similarly, $s \leqslant \frac{1}{2}(q+x)$ and $t \leqslant \frac{1}{2}(q+y)$. So we have

$(r+s)+(s+t)+(t+r) = 2r+2s+2t \leqslant 3q+x+y+z \leqslant 4q$

as required.

We can now state and prove our main theorem.

Theorem 5.11. For any $n \geqslant 1$, a good partial latin square of side n with at most $n+1$ non-empty cells can be completed to a latin square of side n.

Proof. We proceed by induction on n, along the lines explained at the beginning of Section 4. We can assume that exactly $n+1$ cells are non-empty ($n=1$ is trivial!). By Corollary 5.9, the theorem is true for $n \leqslant 5$.

Let P be a good partial latin square of side $n \geqslant 6$ with $n+1$ non-empty cells and assume that the theorem holds for partial latin squares of smaller side. Let P be on symbols $1, \ldots, n$. We must show that P can be completed to a latin square of side n.

Case 1. P, or a conjugate of P, contains a diagonal non-empty cell. We assume that P is chosen among its conjugates so as to have as many diagonal preassignments as possible. Let R be an $r \times s$ subarray, containing all non-diagonal non-empty cells of P, chosen as small as possible (and placed in the top left hand corner). We can assume that $r \geqslant s$, and that all non-empty cells of P occur in R or on the diagonal outside R. By Lemma 5.8, we can assume that the last row and column are empty, and that the symbol n does not occur in P. We also assume that $s > \lfloor \frac{1}{2}(n+1) \rfloor$, by Lemma 5.7.

For all i, $1 \leqslant i \leqslant n$, let f(i) be the number of times the symbol i occurs on the diagonal outside R. Let ℓ be a symbol with $f(\ell) \geqslant 1$. Let P' be obtained from P by deleting the last row and column, and a diagonal preassignment of the symbol ℓ. Then P' is a partial latin square of side n-1 on symbols 1,2,...,n-1 with n non-empty cells. As $n \geqslant 6$, $\lfloor \frac{1}{2}(n+1) \rfloor \geqslant 3$, and if P' is bad then Corollary 5.5 applies to P or one of its conjugates. So we assume that P' is good, and, by the induction hypothesis, we can complete P' to a latin square L' of side $n-1$. By Lemma 3.1 we have, in L':

$$R(i) \geqslant r+s-n+1+f(i) \text{ for all } i \neq \ell,\ 1 \leqslant i \leqslant n-1.$$

For the symbol ℓ we have $R(\ell) \geqslant r+s-n+1+f(\ell)-1 = r+s-n+f(\ell)$.

We now disregard what is outside R, except the diagonal preassignments of P. We want to apply Theorem 3.2 to embed a modified version R_m of R in a latin square L which is a completion of P. To do that we must have

$$R_m(i) \geqslant r+s-n+f(i) \text{ for all } i,\ 1 \leqslant i \leqslant n.$$

This holds for the symbol ℓ with $R_m(\ell) = R(\ell)$. It will hold for any symbol $i \neq \ell$, with $1 \leqslant i \leqslant n-1$, if $R_m(i) \geqslant R(i)-1$. We must make the symbol n not occurring in R occur $r+s-n$ times in R_m.

Suppose that we can find a partial transversal of length $r+s-n$ in R, avoiding all preassigned cells and the symbol ℓ. Then we can place the symbol n in all cells of the partial transversal to obtain R_m. It will then satisfy the inequality for all i, and we can complete by Theorem 3.2.

To find the required partial transversal we apply Lemma 4.2 with the non-empty cells of P in R as forbidden cells and ℓ as the forbidden symbol. There are at most n non-empty cells of P in R; by adding cells arbitrarily we can assume that we have exactly n forbidden cells (there are enough cells to add). Lemma 4.2 then gives the partial transversal we need, if $(n+1-r)\ (n-s) \geqslant n$. By Lemma 4.1, $r+s \leqslant \lfloor \frac{3n}{2} \rfloor$ and so, if $n-r \geqslant 3$ then $s \leqslant \frac{1}{2}(r+s) \leqslant \frac{3n}{4}$, implying

$(n+1-r)\ (n-s) \geqslant 4 \cdot \frac{n}{4} \geqslant n,$

and if $r=n-2$ then $s \leqslant \lfloor \frac{3n}{2} \rfloor - (n-2) = \lfloor \frac{n}{2} \rfloor + 2$, giving

$(n+1-r)\ (n-s) \geqslant 3(\lfloor \frac{n+1}{2} \rfloor - 2) \geqslant n$

for $n \geqslant 11$. Close inspection shows that the condition is satisfied except in the following three cases: (i) $n=10$, $r=8$, $s=7$. (ii) $n=8$, $r=s=6$. (iii) $n=7$, $r=s=5$. In all cases, $r+s > \frac{3(n-1)}{2}$, so P can contain only one diagonal preassignment. It follows that in case (iii) P contains a row with at least 2 non-empty cells such that all other non-empty cells are in at most $\lfloor ½(n+1) \rfloor = 4$ columns, and so P can be completed by Corollary 5.5. Consider cases (i) and (ii). Here $r+s=\frac{3n}{2}$, and it follows from the proof of Lemma 4.1 that each non-empty cell of P is either alone in its row or alone in its column. But as all conjugates of P have at most one diagonal preassignment, at most one of the non-empty cells alone in their rows can have an entry which occurs just once in P, and similarly for the cells alone in their columns. So at most $2+½(n+1-2)$ distinct symbols occur in P. This is $½(n+3)$, so P can be completed by Corollary 5.6.

Case 2. Neither P nor any of its conjugates contains a diagonal non-empty cell. We suppose that P is chosen among its conjugates so as to have $r \geqslant s$ and $r+s \leqslant \frac{4(n+1)}{3}$ (Lemma 5.10), where all non-empty cells are inside the $r \times s$ subarray R having no empty rows or columns. By Corollary 5.6, we can assume $s \geqslant \lfloor ½(n+3) \rfloor + 1$. By the same corollary, we can assume that there is a symbol which is preassigned exactly once in P; by symmetry let it be the symbol 1 in cell (1,1). Let P' be obtained from P by deleting the last row and column and removing the symbol 1 from cell (1,1). Then P' is partial latin square of side $n-1$ on symbols 2,...,n with n non-empty cells. As in Case 1, we see that we can assume that P' is good. By the induction hypothesis, we can complete P' to a latin square L' of side n-1, and we have, in L',

$R(i) \geqslant r+s-n+1$ for all i, $2 \leqslant i \leqslant n$,

and we need to modify R to R_m with

$R_m(i) \geqslant r+s-n$ for all i, $1 \leqslant i \leqslant n$.

Having obtained R_m, we can complete by Theorem 3.2. So what we have to do is make the symbol 1 occur $r+s-n$ times, and we can delete any other symbol once.

Let k be the symbol placed in cell (1,1) of L'. We replace this occurrence of k by the symbol 1. Then we cannot delete any further occurrences of k, and we need $r+s-n-1$ additional occurrences of 1.

We look for a partial transversal of length $r+s-n-1$ in the $(r-1)\times(s-1)$ latin rectangle R' obtained from R by deleting the first row and column. We let the preassigned cells of R' be forbidden cells, and in each column of R' with no preassigned cell we choose an arbitrary cell as a forbidden cell (then the cell of that column which is in R but not in R' must be preassigned in P). Of the $n+1$ non-empty cells of P, at least 2 do not correspond to forbidden cells in R', namely cell (1,1) and some other cell in the first column of R, because if (1,1) were the sole prescribed cell in its column, a conjugate of P would have a diagonal cell (as the symbol 1 was not prescribed anywhere else). So at most $n-1$ cells are forbidden in R'. We may add cells so as to have exactly $n-1$. By Lemma 4.2, we can find the required partial transversal of length $r+s-n-1= (r-1)+(s-1)-(n-1)$ in R', avoiding all forbidden cells and the symbol k, if

$$((n-1)+1-(r-1))\ ((n-1)-(s-1))\geqslant n-1$$

which is

$$(n+1-r)\ (n-s)\geqslant n-1.$$

If $n-r\geqslant 3$ then $s\leqslant \tfrac{1}{2}(r+s)\leqslant \frac{2(n+1)}{3}$ and so $(n+1-r)\ (n-s)\geqslant \frac{4(n-2)}{3}\geqslant n-1$ as $n\geqslant 6$.
If $n-r\leqslant 2$ then $s\leqslant \lfloor\frac{4(n+1)}{3}\rfloor-(n-2)=\lfloor\frac{n+10}{3}\rfloor\leqslant\lfloor\tfrac{1}{2}(n+3)\rfloor$ unless $n=8$ or $n=6$. In the latter case we must have $r=5$ but then we get $s\leqslant 4=\lfloor\tfrac{1}{2}(n+3)\rfloor$. In the former case we get $r=s=6$, and each of the 7 symbols occurs at least 5 times in R (before k is replaced by 1); it follows that exactly one symbol occurs 6 times, say the symbol b. Then b occurs at least 4 times in the 5×5 subsquare (with at most 6 prescribed cells) that we consider for our transversal, and at most 2 of the occurrences can be in prescribed cells (this follows from the proof of Lemma 5.10). Thus we can let b occur twice in our 'transversal' (if $b=k$, b occurs 5 times in the 5×5 subsquare, and we can let the transversal include k). Then it is easy to see that we can find the required 'transversal'.

So in all cases, we can add $r+s-n-1$ occurrences of the symbol 1 and then embed R_m to obtain a completion of P.

This finishes the proof of Theorem 5.11.

We finally mention some recent results and conjectures on completing partial latin squares with no symmetry required, all of which are contained in work at least partly due to R. Häggkvist.

Theorem 5.12. (Chetwynd & Häggkvist 1984). There is a constant $c>10^{-5}$ such that every partial latin square of even side $n>10^7$ in which every row, column and symbol is used at most cn times can be completed to a latin square of side n.

For large n, this improves a previous result stating that completion is possible if $n\equiv 0$ modulo 16 and each row, column and symbol is used at most $2^{-7}\sqrt{n}$ times (Daykin & Häggkvist 1984). Theorem 5.12 is probably far from best possible:

Conjecture A. (Daykin & Häggkvist 1984). A partial latin square of side n in which every row, column and symbol is used at most $\frac{n}{4}$ times can be completed to a latin square of side n.

A related problem is expressed in the following conjecture.

Conjecture B. (Häggkvist 1984a). Let P be a partial latin square of side n in which all non-empty cells lie in an $r\times s$ subarray, and assume that each row is used at most n-r times and that each column is used at most n-s times. Then P can be completed to a latin square of side n.

Häggkvist also proved

Theorem 5.13. (Häggkvist 1984b). If P is a partial latin square of side n in which the non-empty cells are precisely all cells in the first q rows and all cells in the first q columns, and in which the cells common to the first q rows and the first q columns form a latin square of side q, then P can be completed to a latin square of side n.

Theorem 5.12 gives a partial solution to a problem of L. Fuchs, which can be formulated:

Let $n=n_1+n_2+\ldots+n_k$ be a partition of n. When does there exist a latin square of side n with latin subsquares of sides $n_1,n_2,\ldots,n_k$ on mutually disjoint sets of rows, mutually disjoint sets of columns and mutually disjoint sets of symbols? By Theorem 5.12, such a latin square exists if $n_i\leqslant cn$ for all i, $1\leqslant i\leqslant k$, and n is large enough. We refer to the literature for further results on Fuchs' problem (Dénes & Pásztor 1963; Dénes & Keedwell 1974; Heinrich 1984).

6. Completion of partial symmetric Latin Squares

The purpose of this section is mainly to state two results on completion of partial symmetric latin squares which are analogous to the Evans Conjecture. Both results are very recent.

The diagonal of a partial symmetric latin square of side n is called *admissible* if the number of symbols occurring with parity different from n does not exceed the number of empty cells. If n is odd, the diagonal is admissible if and only if all its entries are distinct. If the diagonal is not admissible, then the square cannot be completed to a symmetric latin square of side n. The parity condition of Theorem 3.8 and Corollary 3.9 simply ensures that the diagonal is admissible.

Figure 17 shows some partial symmetric latin squares with admissible diagonal, which cannot be completed to symmetric latin squares of the same side.

Theorem 6.1. (Andersen & Hilton 1985). Let $n \geqslant 3$, and let P be a partial symmetric latin square of side n with admissible diagonal.

If P has less than n non-empty cells, then P can be completed to a symmetric latin square of side n.

If P has exactly n non-empty cells then P can be completed if and only if P is not of the form of any of the squares E1, O1, or O2.

If P has exactly n + 1 non-empty cells then P can be completed if and only if P is neither of the form of any of the squares E1, O1 or O2 with a further cell filled nor of the form of any of E2, E3, O3, 5A or 5B.

The proof of Theorem 6.1 is very long. The general idea is very similar to that of the proof of the main theorem of the last section and so is by induction on n, but there are more complications. In particular, the case where all or all but one of the rows are used is very elaborate. When that is done, it is possible to delete a symmetric pair of entries from the square of side n to be completed so as to obtain a partial symmetric latin square of side n – 2 (with the same parity as n). We complete by the induction hypothesis and focus on the latin rectangle R of Figure 6. By Lemma 4.1, we know something about the side of R. We add occurrences of two new symbols by applying Lemma 4.3, and we embed by Theorem 3.8 to obtain the required completion.

Theorem 6.1 can be used to give results on completions of edge-colourings of K_n with the colours of some edges prescribed. Below we state two such results, one for even n and one for odd n. The odd case is

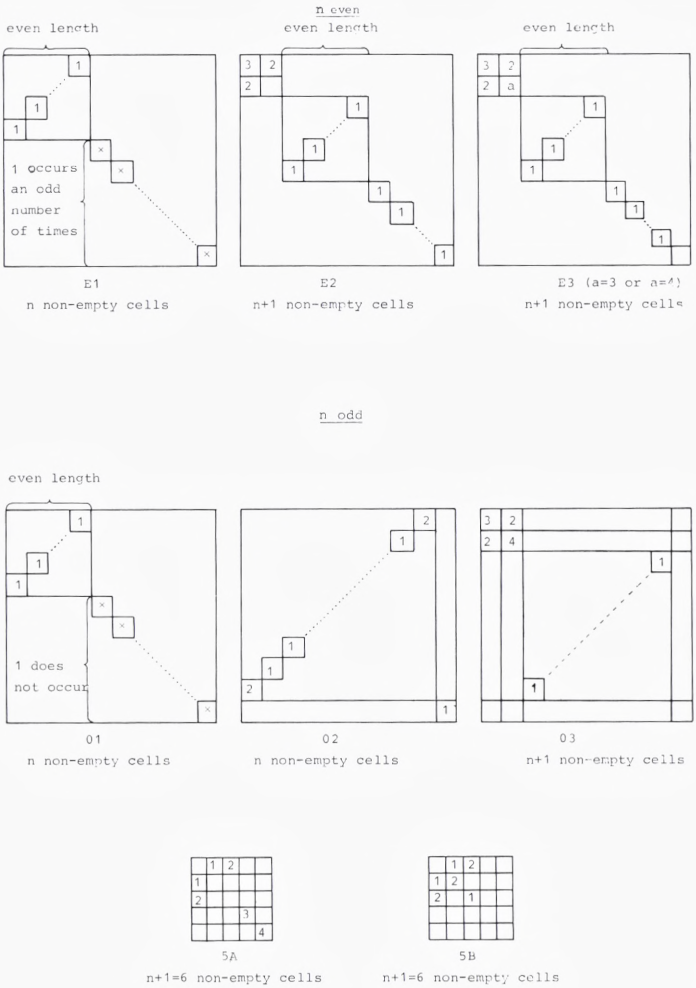


Fig. 17

an easy consequence of the even case, and the even case follows from Theorem 6.1 for odd n alone. The two results are not the strongest possible corollaries of Theorem 6.1 in this direction.

Corollary 6.2. Let C be a set of edges of K_{2m}, $m \geqslant 3$, and assume that the subgraph spanned by the edges of C has an edge-colouring. Then:

a) If $|C| \leqslant m-1$, then the edge-colouring can be extended to an edge-colouring of K_{2m} with $2m-1$ colours.

b) If $|C| = m$ then the edge-colouring can be extended to an edge-colouring of K_{2m} with $2m-1$ colours if and only if the edge-coloured subgraph is not of Type 1 of Figure 18.

c) If $|C| = m+1$ then the edge-colouring can be extended to an edge-

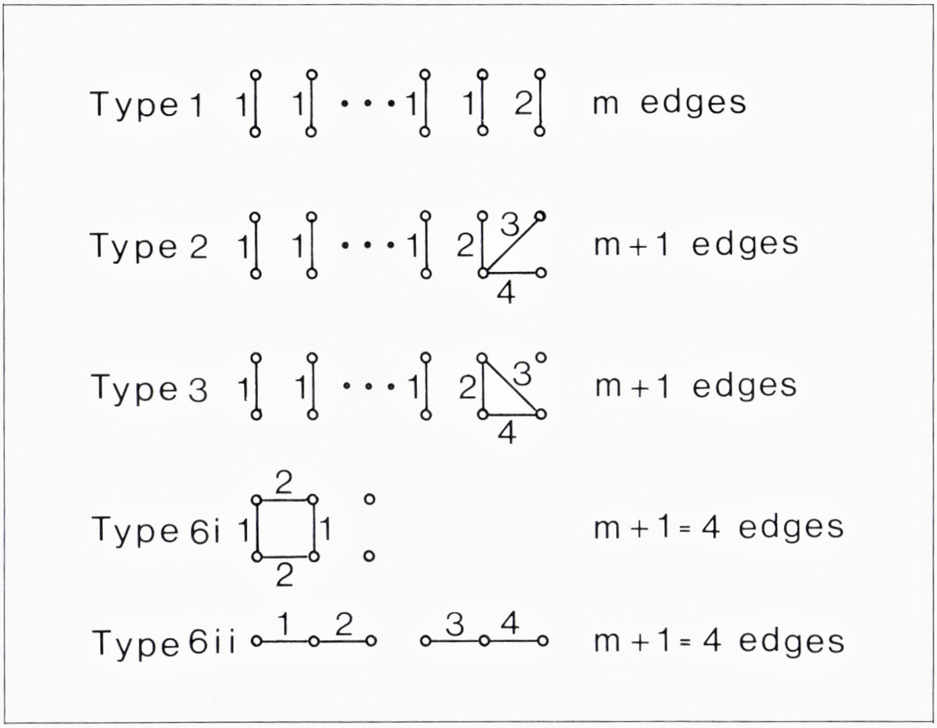


Figure 18

colouring of K_{2m} with $2m-1$ colours if and only if the edge-coloured subgraph is neither of Type 1 with an edge added nor of Type 2, 3, 6i or 6ii of Figure 18.

Corollary 6.3. Let C be a set of edges of K_{2m-1}, $m \geqslant 3$, and assume that the subgraph spanned by the edges of C has an edge-colouring. Then:

a) If $|C| \leqslant m$ then the edge-colouring can be extended to an edge-colouring of K_{2m-1} with $2m-1$ colours.

b) If $|C| = m+1$ then the edge-colouring can be extended to an edge-colouring of K_{2m-1} with $2m-1$ colours if and only if the edge-coloured subgraph is not of Type 4 or 5 of Figure 19.

The next result that we state is more related to Corollaries 6.2 and 6.3 than to Theorem 6.1, as it is concerned with edge-colourings of complete graphs, where each colour is prescribed *at most once*. The theorem was obtained by E. Mendelsohn and the author, and a formulation compatible with the statements above is the following.

Theorem 6.4. (Andersen & Mendelsohn 1985). Let D be a set of edges of K_n with at most $q(K_n)-1$ edges. Then K_n has an edge-colouring with $q(K_n)$ colours so that all edges of D have distinct colours, except if n is even and D is the edge-set of the graph H_{2m} of Figure 20, or if n=6 and D is the edge-set of H_5' or H_6', or if n = 5 and D is the edge-set of H_5'.

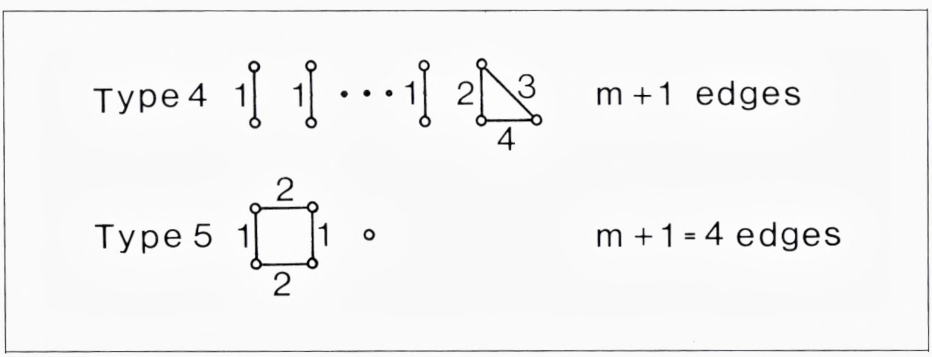
Theorem 6.4 for odd n follows from the result for even n. We also state a reformulation of the even case which stresses that it is a result about 1-factorizations of the complete graph. A *1-factorization* of a graph G = (V,E) is a decomposition of E into mutually disjoint 1-factors. Especially 1-factorizations of the complete graph K_{2m} of even order have been studied extensively (Mendelsohn & Rosa 1984).

Corollary 6.5. Let D be a set of edges of K_{2m}, and let $|D| \leq 2m-2$. Then K_{2m} has a 1-factorization with all edges of D in distinct 1-factors if and only if D is not the edge-set of the graph H_{2m}, or, if m = 3, of H_5' or H_6'.

If most edges of D are concentrated in a 'small' subgraph K_r of K_{2m}, corresponding to R of Figure 6 not being too large, then Corollary 6.5 is proved in the same way as Theorem 6.1.

If not, the proof is completely different (although both cases are treated within the same induction proof); if R is large, the proof relies on a lemma saying that then the vertices of K_{2m} can be split into two sets of m vertices each, so that exactly m or m+1 edges of D join a vertex from one class to a vertex of the other. Then Theorem 5.11 is used on the $K_{m,m}$ formed in this way.

Figure 19



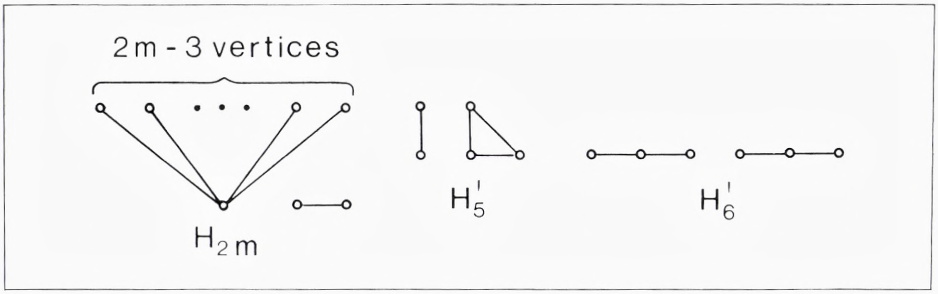


Figure 20

Conjecture C. Let $m \geqslant 5$, and let D be a set of edges of K_{2m} with $|D| \leqslant 2m-1$. Then K_{2m} has a 1-factorization with all edges of D in distinct 1-factors, if and only if

(i) D does not contain the edge-set of H_{2m}, and

(ii) K_{2m} does not have two distinct vertices U and V for which $UV \notin D$ but each edge of D is incident with either U or V, $|D| = 2m-1$.

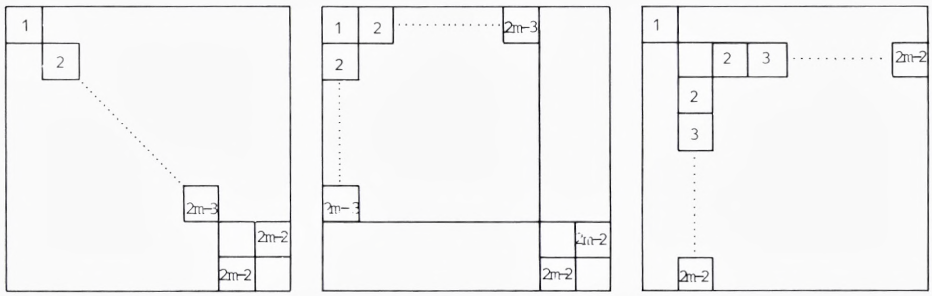
For $m \leqslant 4$, there are several exceptions to Conjecture C.

The work on Theorem 6.4 was to a large extent initiated by a paper by A. Hartman on partial triple systems and edge-colourings (Hartman 1984). It has some consequences for completions of partial symmetric latin squares, supplementing Theorem 6.1.

If we define an *appearance* of a symbol in a partial symmetric latin square as either an occurrence in a diagonal cell or two occurrences in a symmetric pair of cells, then n non-empty cells may correspond to as little as $\frac{n}{2}$ appearances. In the case where no symbol *appears* more than once, we can strengthen Theorem 6.1 by applying Corollary 6.5.

Corollary 6.6. Let P be a partial symmetric latin square of odd side $2m-1$ in which one symbol does not appear and each of the remaining $2m-2$ symbols has at most one appearance. Then P can be completed to a symmetric latin square of side $2m-1$ if and only if P is not of the form of any of the squares of Figure 21.

Corollary 6.7. Let P be a partial symmetric latin square of even side 2m in which one symbol does not appear and each of the remaining $2m-1$ symbols has at most one appearance, one of them not appearing outside



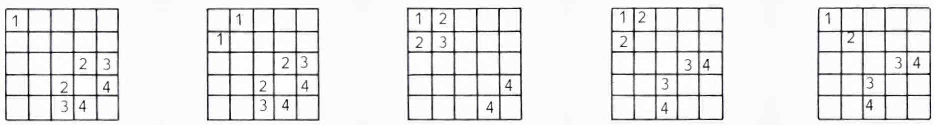


Figure 21

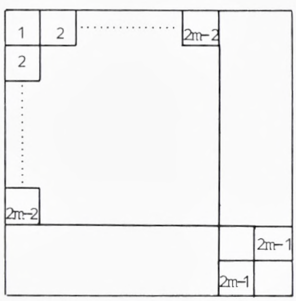


Figure 22

the diagonal. Then P can be completed to a symmetric latin square of side 2m if and only if P is not of the form of the square in Figure 22, and at most m symbols occur on the diagonal.

7. Final remarks

Many topics and results that would fit in well with the title of this paper have not been included. And we do not even claim to have covered the most *important* subjects. The choice of material has been as much affected by the author's personal preferences as by any assignment of different levels of importance to the topics. It has also been a wish to make the contents coherent rather than desultory.

For example, we have not mentioned the word *quasigroup* at all, and yet it a concept almost identical to that of a latin square.

A quasigroup (Q,⋆) is a set Q with an operation ,⋆, such that for all a and b in Q, each of the equations $a \star x = b$ and $x \star a = b$ is uniquely solvable in x. A latin square is the same as a multiplication table for a quasigroup. We have imposed very little extra structure on our latin squares in this paper, basically only symmetry ($x \star y = y \star x$) and idempotency ($x \star x = x$). If the quasigroup is required to satisfy other simple identities, further interesting completion problems arise (one of the more famous problems among these is that of completing partial Steiner triple systems). We refer the reader to the literature (Lindner 1984).

The book which is the standard reference on latin squares is that by J. Dénes and A. D. Keedwell. It emphasizes the quasigroup point of view and contains many references (Dénes & Keedwell 1974). At the time of writing, Dénes and Keedwell are editing a new, comprehensive volume on latin squares (two of the references that we have given are to manuscripts written for this volume) (Dénes and Keedwell 1986?).

The present paper is meant to have two purposes: Partly to survey the area of completing partial latin squares, and partly to announce some new results in that area, carrying out the details of proof for one of these. We hope that the reader has realized that such completion problems, even though they are often very easy to formulate, can be quite intricate. So it appears to be a fitting end to this paper to ask: Just *how* intricate is the problem of completing partial latin squares?

We can define the intricacy of completing partial latin squares as the least integer k satisfying the following:

For any integer n, any partial latin square of side n can be partitioned into k partial latin squares of side n each of which can be completed to a latin square of side n.

Partitioning a partial latin square P into $P_1, \ldots, P_k$ means filling some cells of the P_t's such that if cell (i,j) of P is non-empty, then its entry occurs in cell (i,j) of one of the P_t's, and all non-empty cells of the P_t's are obtained in this way.

D. E. Daykin and R. Häggkvist posed the problem of showing that the intricacy of completing partial latin squares is 2. The concept of intricacy was later generalized to a large class of combinatorial construction problems (Daykin and Häggkvist 1981; Daykin and Häggkvist 1984; W. E. Opencomb 1984).

It follows from Corollary 3.3 (Corollary 3.4 is enough if n is even) that there exists a finite k satisfying the condition of the definition of intricacy, because k=4 will do. So the question is whether the intricacy is 2, 3 or 4.

Conjecture D. The intricacy of completing partial latin squares is 2.

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